Iwasawa-Tate on ζ-functions and L-functions

After the main part, namely, *analytic continuation* and *functional equation* of global zeta integrals:

- [recap] The elementary *global* integral is

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, dx = \frac{|\mathbb{J}^1/k^\times|}{s-1} \qquad \text{(for } \mathrm{Re}(s) > 1\text{)}$$

- [done] Vanishing of ramified elementary integrals
- [recap] Good finite-prime *local* zeta integrals

$$\int_{k_v^\times \cap \mathfrak{o}_v} |x|^s\, dx = \frac{1}{1-q_v^{-s}} \qquad \text{(for } \mathrm{Re}(s) > 0\text{)}$$

- *Local* functional equations

$$\frac{Z_v(s, f_v)}{Z_v(1-s, \widehat{f}_v)} = \frac{Z_v(s, g_v)}{Z_v(1-s, \widehat{g}_v)} \qquad \text{(for } f, g \in \mathscr{S}(k_v)\text{)}$$

- Unramified and ramified archimedean zeta integrals

The elementary global integral :

The poles and residues of zeta integrals are multiples of an elementary integral over $\mathbb{J}^+/k^\times$, which we claim is

$$\int_{\mathbb{J}^+/k^\times} |x|^{1-s}\, d^\times x = \frac{|\mathbb{J}^1/k^\times|}{s-1}$$

Multiplicative measures on $\mathbb{J}$ and $k_v^\times$ are completely determined by giving local units $\mathfrak{o}_v^\times$ measure 1 at *all* finite places, and $d^\times x = \frac{d^+ x}{|x|_v}$ at archimedean places.

For *abelian* (hence, *unimodular*) topological groups, the general riff

$$\int_G f(g)\, dg \;=\; \int_{H\backslash G} \Big(\int_H f(h\dot{g})\, dh\Big)\, d\dot{g}$$

applies: fixing any two of the three measures uniquely specifies the normalizing constant for the third so that the equation holds.

Measures on $k_v^\times$ specifies the measure on $\mathbb{J}$. Counting measure on $k^\times$ uniquely specifies the measure on $\mathbb{J}/k^\times$ by the above identity

$$\int_{\mathbb{J}} f(g)\,dg \;=\; \int_{\mathbb{J}/k^\times} \sum_{h \in k^\times} f(h\dot{g})\,d\dot{g}$$

Since $\mathbb{J}^1$ is the kernel of $|\cdot|$, $\mathbb{J}^1/k^\times$ fits into an exact sequence

$$1 \longrightarrow \mathbb{J}^1/k^\times \longrightarrow \mathbb{J}/k^\times \longrightarrow \mathbb{R}^+ \longrightarrow 1 \qquad (\mathbb{R}^+ = (0, +\infty))$$

Thus, the usual measure $\frac{dx}{x}$ on $\mathbb{R}^+$ and the measure on $\mathbb{J}/k^\times$ uniquely determine the measure on $\mathbb{J}^1/k^\times$ by

$$\int_{\mathbb{J}/k^\times} f(g)\,dg \;=\; \int_{(\mathbb{J}/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} f(h\dot{g})\,dh \Big)\,d\dot{g}$$

$$= \int_{\mathbb{R}^+} \Big(\int_{\mathbb{J}^1/k^\times} f(h\dot{g})\,dh \Big)\,d\dot{g}$$

It is not necessary, but it is easy to identify a *section* $\sigma : \mathbb{R}^+ \to \mathbb{J}$ with

$$|\sigma(t)| \;=\; t$$

For $k = \mathbb{Q}$, just map $t \to (t, 1, 1, \ldots)$, the idele with trivial entries except at $\mathbb{Q}_\infty^\times \approx \mathbb{R}^\times$, where the entry is t. For general number fields k, with r_1, r_2 real-and-complex completions, let

$$\sigma(t) \;=\; (t^{\frac{1}{r_1+r_2}}, \ldots, t^{\frac{1}{r_1+r_2}}, 1, 1, 1, 1, \ldots)$$

with non-trivial entries at archimedean places.

With f being the product of $|\cdot|^{1-s}$ and the characteristic function of $\mathbb{J}^+/k^\times$, this gives

$$\int_{\mathbb{J}^+/k^\times} |g|^{1-s}\,dg \;=\; \int_{(\mathbb{J}^+/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} |gh|^{1-s}\,dh\Big)\,d\dot{g}$$

$$= \int_{(\mathbb{J}^+/k^\times)/(\mathbb{J}^1/k^\times)} \Big(\int_{\mathbb{J}^1/k^\times} |g|^{1-s}\,dh\Big)\,d\dot{g}$$

$$= \int_{[1,+\infty)} |\dot{g}|^{1-s}\Big(\int_{\mathbb{J}^1/k^\times} 1\,dh\Big)\,d\dot{g} \;=\; |\mathbb{J}^1/k^\times|\cdot\int_1^\infty t^{1-s}\,\frac{dt}{t}$$

$$= |\mathbb{J}^1/k^\times|\cdot\int_1^\infty t^{-s}\,dt \;=\; |\mathbb{J}^1/k^\times|\cdot\Big[\frac{t^{1-s}}{1-s}\Big]_1^\infty \;=\; \frac{|\mathbb{J}^1/k^\times|}{s-1} \qquad ///$$

Remark: *Postpone* the non-elementary computation that

$$|\mathbb{J}^1/k^\times| \;=\; \frac{2^{r_1}\,(2\pi)^{r_2}\,h\,R}{D_k^{\frac{1}{2}}\,w}$$

Good finite-prime local integrals: $\int_{k_v^\times \cap \mathfrak{o}_v} |x|_v^s \, d^\times x$

Good includes the assertion that the local Schwartz function f_v in the local zeta integral is the *characteristic function* of the local integers $\mathfrak{o}_v$.

By convention, *archimedean* primes are *never* good.

The good prime assumption includes the assertion that k_v is *absolutely unramified*, meaning k_v is unramified over the corresponding completion $\mathbb{Q}_p$, meaning p *stays prime* in $\mathfrak{o}_v$.

We will show that unramifiedness entails that the natural measure is $|\mathfrak{o}_v| = 1$, and the Fourier transform of the characteristic function of $\mathfrak{o}_v$ is *itself*. These points do not affect the local *multiplicative* computation.

At finite primes, the multiplicative Haar measure is always normalized so that $|\mathfrak{o}_v^\times| = 1$. Then the usual

$$\int_G f(g)\,dg \;=\; \int_{G/H}\int_H f(\dot{g}h)\;dh\;\;dg$$

with f the product of $|\cdot|_v^s$ and the characteristic function of $\mathfrak{o}_v$ gives

$$\int_{k_v^\times} f(g)\,dg \;=\; \int_{k_v^\times/\mathfrak{o}_v^\times}\int_{\mathfrak{o}_v^\times} f(\dot{g}h)\;dh\;\;dg$$

$$= \int_{(k_v^\times\cap\mathfrak{o}_v)/\mathfrak{o}_v^\times}\int_{\mathfrak{o}_v^\times} |\dot{g}h|_v^s\;dh\;\;dg \;=\; \int_{(k_v^\times\cap\mathfrak{o}_v)/\mathfrak{o}_v^\times} |\dot{g}|_v^s\Big(\int_{\mathfrak{o}_v^\times} 1\;dh\Big)\;dg$$

$$= \int_{(k_v^\times\cap\mathfrak{o}_v)/\mathfrak{o}_v^\times} |\dot{g}|_v^s\;dg \;=\; \sum_{n=0}^{\infty} |p^n|_v^s \;=\; \frac{1}{1-|p|_v^{-s}} \;=\; \frac{1}{1-q_v^{-s}}$$

where $q_v \;=\; |p|_v^{-1}$ is the residue field cardinality. ///

The same computation applies to the *seemingly* more general

$$Z_v(s, \chi_v, f_v) \;=\; \int_{k_v^\times} |x|_v^s \, \chi_v(x) \, f_v(x) \; d^\times x$$

with f_v the characteristic function of $\mathfrak{o}_v$ and χ_v *unramified*, meaning that χ_v is trivial on $\mathfrak{o}_v^\times$. That is, the group homomorphism χ_v is $\mathfrak{o}_v$-invariant, so is inescapably of the form

$$\chi_v(x) \;=\; |x|_v^{it_\chi} \qquad\qquad \text{(for some } t_\chi \in \mathbb{R} \text{ depending on } \chi_v)$$

Then the unramified non-archimedean local zeta factor is

$$Z_v(s, \chi_v, f_v) \;=\; \int_{k_v^\times} |x|_v^s \, \chi_v(x) \, f_v(x) \; d^\times x$$

$$= \int_{k_v^\times} |x|_v^{s+it_\chi} \, f_v(x) \; d^\times x \;=\; \frac{1}{1 - q_v^{-s-it_\chi}}$$

This kind of shifting occurs for all kinds of L-functions...

For example, for groundfield $k = \mathbb{Q}$, for Dirichlet L-functions $L(s, \chi)$ the good-prime factors are

$$\frac{1}{1 - \frac{\chi(p)}{p^s}} = \frac{1}{1 - p^{-s + \frac{i\theta_p}{\log p}}} \qquad \text{(where } \chi(p) = e^{i\theta_p}\text{)}$$

That is, here the local characters at unramified $p \sim v$ are

$$\chi_v(x) = |x|_v^{-\frac{i\theta_p}{\log p}}$$

with $e^{i\theta_p}$ a root of unity. For an *ideal class character* χ, for number field k, for local parameter ϖ_v in k_v,

$$\chi_v(\varpi_v)^h = 1 \qquad \text{(with } h = h(\mathfrak{o})\text{)}$$

so $\chi_v(x_v) = |x|_v^{\frac{2\pi i \ell}{h \log q_v}}$ for some $\ell \in \mathbb{Z}$.

For general *größencharakteren* there is no connection to roots of unity.

Local functional equations: For $0 < \mathrm{Re}(s) < 1$, all the local zeta integrals in the following are absolutely convergent, and

$$\frac{Z_v(s, f_v)}{Z_v(1-s, \widehat{f}_v)} = \frac{Z_v(s, g_v)}{Z(1-s, \widehat{g}_v)} \qquad \text{(for } f, g \in \mathscr{S}(k_v))$$

We also want the version of this for *ramified* characters.

The first point of this *local functional equation* is to prove that local zeta integrals with any Schwartz functions whatsoever are meromorphic, granting that *one* (non-zero) local zeta integral at v is meromorphic.

Proof: Postpone the convergence argument. Take $0 < \mathrm{Re}(s) < 1$, so local zeta integrals for both $Z(s, f)$ and $Z(1-s, \widehat{g})$ converge absolutely. Then the local functional equation is a direct computation: *expand the definition and change variables...*

Suppress all the subscripts v! Replace y by yx/η in

$$Z(s,f)\,Z(1-s,\widehat{g}) \;=\; \int_{k^\times}\int_{k^\times} |x|^s\, f(x)\; |y|^{1-s}\, \widehat{g}(y)\; d^\times x\; d^\times y$$

$$= \int_{k^\times}\int_{k^\times}\int_{k} \overline{\psi}(y\eta)\, |x|^s\, f(x)\; |y|^{1-s}\, g(\eta)\; d^+\eta\; d^\times y\; d^\times x$$

$$= \int_{k}\int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\, |x|^s\, f(x)\; |yx/\eta|^{1-s}\, g(\eta)\; d^\times y\; d^\times x\; d^+\eta$$

$$= \int_{k}\int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\, |x|^1\, f(x)\; |y|^{1-s}\, |\eta|^{s-1}\, g(\eta)\; d^\times y\; d^\times x\; d^+\eta$$

The measure $|x|\cdot d^\times x$ is a constant multiple of the *additive* Haar measure d^+x. The precise constant is irrelevant, since it cancels itself in the necessary rearrangement:

$$|x|\,|\eta|^{-1}\; d^\times x\; d^+\eta \;=\; d^+x\; d^\times\eta$$

Remark: In general, there is no compulsion to superscript the multiplicative and additive Haar measures, but here the change back-and-forth makes this necessary.

It is convenient that for local fields k and $k^\times$ differ by a single point, of additive measure 0. Thus, continuing,

$$Z(s,f)\,Z(1-s,\widehat{g})$$

$$= \int_{k^\times}\int_k\int_{k^\times} \overline{\psi}(yx)\,f(x)\,|y|^{1-s}\,|\eta|^s\,g(\eta)\;d^\times y\;d^+x\;d^\times\eta$$

$$= \int_{k^\times}\int_{k^\times} \widehat{f}(y)\,|y|^{1-s}\,|\eta|^s\,g(\eta)\;d^\times y\;d^\times\eta \;=\; Z(1-s,\widehat{f})\,Z(s,g)$$

This proves the local functional equation in $0 < \mathrm{Re}(s) < 1$. The general assertion follows from the Identity Principle. ///

Remark: The local functional equations do not yield the *global* functional equation! They only prove the essential irrelevance of the Schwartz data.

Real zeta integrals:

Although archimedean places are never *good*, they are tractable. The standard *unramified* local integral for $v \approx \mathbb{R}$ uses the Gaussian $f(x) = e^{-\pi x^2}$:

$$Z_{\mathbb{R}}(s, e^{-\pi x^2}) = \int_{\mathbb{R}^\times} |y|^s \, e^{-\pi y^2} \, \frac{dy}{|y|} = 2\int_0^\infty |y|^s \, e^{-\pi y^2} \, \frac{dy}{y}$$

$$= \int_0^\infty |y|^{\frac{s}{2}} \, e^{-\pi y} \, \frac{dy}{y} \qquad \text{(replacing } y \text{ by } \sqrt{y})$$

$$= \pi^{-\frac{s}{2}} \int_0^\infty |y|^{\frac{s}{2}} \, e^{-y} \, \frac{dy}{y} = \pi^{-\frac{s}{2}} \cdot \Gamma\big(\frac{s}{2}\big)$$

Remark: This recovers Riemann's gamma factor, despite the integral starting out with a different-looking normalization than Riemann's integral representation

$$\int_0^\infty y^{\frac{s}{2}} \, \frac{\theta(iy) - 1}{2} \, \frac{dy}{y}$$

The only *ramified* character on $\mathbb{R}^\times$ is $y \to \mathrm{sgn}(y)|y|^s$.

The standard *ramified* local integral for $v \approx \mathbb{R}$ uses $f(x) = xe^{-\pi x^2}$:

$$Z_{\mathbb{R}}(s, \mathrm{sgn}, xe^{-\pi x^2}) \;=\; \int_{\mathbb{R}^\times} |y|^s \; \mathrm{sgn}(y) \cdot ye^{-\pi y^2} \; \frac{dy}{|y|}$$

$$= \int_{\mathbb{R}^\times} |y|^s \cdot |y| \cdot e^{-\pi y^2} \; \frac{dy}{|y|} \;=\; 2\int_0^\infty |y|^{s+1} \; e^{-\pi y^2} \; \frac{dy}{y}$$

$$= \int_0^\infty |y|^{\frac{s+1}{2}} \; e^{-\pi y} \; \frac{dy}{y} \qquad\qquad \text{(replacing } y \text{ by } \sqrt{y}\text{)}$$

$$= \pi^{-\frac{s+1}{2}} \int_0^\infty |y|^{\frac{s+1}{2}} \; e^{-y} \; \frac{dy}{y} \;=\; \pi^{-\frac{s+1}{2}} \; \Gamma(\frac{s+1}{2})$$

This recovers the gamma factor for *odd* Dirichlet character L-functions $L(s, \chi)$, for example.

Complex zeta integrals

The correct normalization of measure, norm, and Fourier transform on $k_v \approx \mathbb{C}$ require some attention. This is typical of non-archimedean extensions $k_v/\mathbb{Q}_p$, too, but we have less prejudice about computations there than on $\mathbb{C}$.

Again, the product formula requires

$$|z|_{\mathbb{C}} \;=\; |N^{\mathbb{C}}_{\mathbb{R}}(z)|_{\mathbb{R}} \;=\; |z|^2 \qquad\qquad \text{(the latter the } \textit{usual} \text{ norm)}$$

That is, our *usual* norm is the *extension* from $\mathbb{R}$ to $\mathbb{C}$, while the product formula demands something else, namely, *composition with Galois norm.*

Similarly, the *additive character* $\psi_{\mathbb{C}}(z)$ is

$$\psi_{\mathbb{C}}(z) \;=\; \psi_{\mathbb{R}}(\mathrm{tr}^{\mathbb{C}}_{\mathbb{R}}(z)) \;=\; e^{2\pi i(z+\overline{z})} \;=\; e^{4\pi i \mathrm{Re}\,(z)}$$

Since we cannot talke about *ramification of primes*, nor local or global *differents* $\mathfrak{d}_v, \mathfrak{d}$ as for non-archimedean places, suitable normalization of measure on $k_v \approx \mathbb{C}$ is determined by choice of character and the requirement that *Fourier inversion* hold with the same measure on both copies of $\mathbb{C}$, the original as well as the spectral side.

That is, determine a measure constant c by requiring, for all $f \in \mathscr{S}(\mathbb{C})$,

$$f(z) \;=\; \int_{\mathbb{C}} \int_{\mathbb{C}} \psi_{\mathbb{C}}(zw)\, \psi_{\mathbb{C}}(-w\zeta)\, f(\zeta)\, c\, d\zeta\, c\, dw$$

That is, letting $z = x + iy$, $w = u + iv$, and $\zeta = \xi + i\eta$,

$$f(z) \;=\; c^2 \cdot \int_{\mathbb{C}} \int_{\mathbb{C}} e^{2\pi i \mathrm{tr}(w(z-\zeta))}\, f(\zeta)\, d\zeta\, dw$$

$$=\; c^2 \cdot \int_{\mathbb{C}} \int_{\mathbb{C}} e^{2\pi i((2u)(x-\xi)+(-2v)(y-\eta))}\, f(\zeta)\, d\zeta\, dw$$

We know Fourier inversion holds with the *usual* measure on $\mathbb{C}$, and with character $e^{2\pi i(ux+vy)}$. To compare to this, in the integral above replace u by $u/2$ and v by $-v/2$, giving

$$\frac{c^2}{2^2} \cdot \int_{\mathbb{C}} \int_{\mathbb{C}} e^{2\pi i(u(x-\xi)+v(y-\eta))} \, f(\zeta) \, d\zeta \, dw \;=\; \frac{c^2}{2^2} \cdot f(z)$$

Thus, the proper normalization of measure on $\mathbb{C}$ for Iwasawa-Tate is

$$d_{\mathbb{C}}(z) \;=\; 2 \cdot d_{\text{usual}}(z)$$

Next, determine a Gaussian $e^{-c(x^2+y^2)}$ which is its own Fourier transform.

$$2 \cdot \int_{\mathbb{C}} e^{-4\pi i(ux-vy)} \, e^{-c(x^2+y^2)} \, dx \, dy$$

$$= \frac{2\pi}{c} \cdot \int_{\mathbb{C}} e^{-2\pi i((2u)x-(2v)y)} \, e^{-\pi(x^2+y^2)} \, dx \, dy \;=\; \frac{2\pi}{c} \cdot e^{-\frac{4\pi^2}{c}(u^2+v^2)}$$

Thus, two reasons for $c = 2\pi$, so $f(w) = e^{-2\pi w\overline{w}}$.

Try taking corresponding multiplicative measure $2\,dz/|z|_{\mathbb{C}}$. Thus, the standard *unramified* complex zeta integral is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s \, e^{-2\pi z\overline{z}} \, \frac{2\,dz}{|z|_{\mathbb{C}}} \; = \; 4\pi \int_0^\infty r^{2s} \, e^{-2\pi r^2} \, r \, \frac{dr}{r^2}$$

$$= \; 4\pi \int_0^\infty r^{2s} \, e^{-2\pi r^2} \, \frac{dr}{r} \; = \; 2\pi \int_0^\infty r^s \, e^{-2\pi r} \, \frac{dr}{r} \; = \; 2\pi \cdot (2\pi)^{-s} \, \Gamma(s)$$

The extra constant 2π in front (*not* the $(2\pi)^{-s}$) suggests renormalizing the multiplicative measure by dividing through by 2π. Some sources do this, others leave the extra 2π.

The *ramified* unitary characters of $\mathbb{C}^\times$ are

$$\chi_\ell(re^{i\theta})|re^{i\theta}|_{\mathbb{C}}^s \;=\; e^{i\ell\theta}\cdot r^{2s} \qquad\qquad (\text{for } \ell\in\mathbb{Z})$$

The standard choice of Schwartz function for the complex zeta integral depends on the sign of $\ell\in\mathbb{Z}$. For $\ell\geq 0$, it is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s\;\chi_\ell(z)\;\overline{z}^\ell\;e^{-2\pi z\overline{z}}\;\frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$=\;\int_0^{2\pi}\int_0^\infty r^{2s}\;e^{i\ell\theta}\;(re^{-i\theta})^\ell\;e^{-2\pi z\overline{z}}\;\frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$=\;4\pi\int_0^\infty r^{2s+\ell}\;e^{-2\pi r^2}\;r\,\frac{dr}{r^2}\;=\;4\pi\int_0^\infty r^{2s+\ell}\;e^{-2\pi r^2}\;\frac{dr}{r}$$

$$=\;2\pi\int_0^\infty r^{s+\frac{\ell}{2}}\;e^{-2\pi r}\;\frac{dr}{r}\;=\;2\pi\cdot(2\pi)^{-(s+\frac{\ell}{2})}\int_0^\infty r^{s+\frac{\ell}{2}}\;e^{-r}\;\frac{dr}{r}$$

$$=\;2\pi\cdot(2\pi)^{-(s+\frac{\ell}{2})}\,\Gamma\big(s+\frac{\ell}{2}\big) \qquad\qquad (\text{for } 0\leq\ell\in\mathbb{Z})$$

For $\ell \le 0$, the standard ramified complex local zeta integral is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s \;\; \chi_\ell(z) \;\; z^{|\ell|} \;\; e^{-2\pi z\overline{z}} \; \frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$= \int_0^{2\pi} \int_0^\infty r^{2s} \; e^{i\ell\theta} \; (re^{i\theta})^{-\ell} \; e^{-2\pi z\overline{z}} \; \frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$= \; 4\pi \int_0^\infty r^{2s-\ell} \; e^{-2\pi r^2} \; r \, \frac{dr}{r^2} \; = \; 4\pi \int_0^\infty r^{2s-\ell} \; e^{-2\pi r^2} \; \frac{dr}{r}$$

$$= \; 2\pi \int_0^\infty r^{s-\frac{\ell}{2}} \; e^{-2\pi r} \; \frac{dr}{r} \; = \; 2\pi \cdot (2\pi)^{-(s-\frac{\ell}{2})} \int_0^\infty r^{s-\frac{\ell}{2}} \; e^{-r} \; \frac{dr}{r}$$

$$= \; 2\pi \cdot (2\pi)^{-(s-\frac{\ell}{2})} \, \Gamma\big(s - \frac{\ell}{2}\big) \qquad\qquad (\text{for } \ell \le 0)$$

Thus, for *both* $\ell \ge 0$ and $\ell \le 0$, the local zeta integral is

$$2\pi \cdot (2\pi)^{-(s+\frac{|\ell|}{2})} \, \Gamma\big(s + \frac{|\ell|}{2}\big) \qquad\qquad (\text{for both } \ell \ge 0 \text{ and } \ell \le 0)$$