**Iwasawa-Tate** on $\zeta$-functions and $L$-functions

*After* the main part, namely, *analytic continuation* and *functional equation* of global zeta integrals...

- [recap] Local functional equations

$$\frac{Z_v(s, f_v)}{Z_v(1-s, \widehat{f}_v)} = \frac{Z_v(s, g_v)}{Z_v(1-s, \widehat{g}_v)} \qquad \text{(for } f, g \in \mathscr{S}(k_v))$$

- [recap] Unramified and ramified archimedean zeta integrals
- Convergence of local zeta integrals
- Archimedean Fourier transform eigenfunctions (Hecke's identity)

**Local functional equations:** For $0 < \mathrm{Re}(s) < 1$, the local zeta integrals are absolutely convergent, and

$$\frac{Z_v(s,\chi,f_v)}{Z_v(1-s,\overline{\chi},\widehat{f}_v)} = \frac{Z_v(s,\chi,g_v)}{Z(1-s,\overline{\chi},\widehat{g}_v)} \qquad (\text{for } f,g \in \mathscr{S}(k_v))$$

The first point is that local zeta integrals with any Schwartz functions whatsoever are meromorphic, granting that *one* (non-zero) local zeta integral at $v$ is meromorphic:

$$Z_v(1-s,\overline{\chi},\widehat{g}) = \frac{Z_v(1-s,\overline{\chi},\overline{f}) \cdot Z_v(s,\chi,g)}{Z_v(s,\chi,f)}$$

gives the meromorphic continuation of $Z_v(s,\widehat{g})$, etc.

**Remark:** The *local* functional equations do not yield the *global* functional equation! They only prove the essential irrelevance of the Schwartz data.

*Proof:* [Recap with $\chi$...] Take $0 < \mathrm{Re}(s) < 1$, so local zeta integrals for both $Z(s,\chi,f)$ and $Z(1-s,\overline{\chi},\widehat{g})$ converge absolutely. Direct computation: *expand the definition and change variables...*

Suppress all subscripts.

$$Z(s,\chi,f)\,Z(1-s,\overline{\chi},\widehat{g})$$

$$= \int_{k^\times}\int_{k^\times} |x|^s\,\chi(x)\,f(x)\,|y|^{1-s}\,\overline{\chi}(y)\,\widehat{g}(y)\;d^\times x\;d^\times y$$

$$= \int_{k^\times}\int_{k^\times}\int_k \overline{\psi}(y\eta)\,|x|^s\,\chi(x)\,f(x)\,|y|^{1-s}\,\overline{\chi}(y)\,g(\eta)\;d^+\eta\;d^\times y\;d^\times x$$

Replacing $y$ by $\frac{yx}{\eta}$, this is

$$\int_k\int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\,|x|^s f(x)\left|\frac{yx}{\eta}\right|^{1-s}\overline{\chi}(y)\chi(\eta)g(\eta)\,d^\times y\,d^\times x\,d^+\eta$$

$$= \int_k\int_{k^\times}\int_{k^\times} \overline{\psi}(yx)\,|x|^1 f(x)\,\overline{\chi}(y)\,|y|^{1-s}\,\chi(\eta)|\eta|^{s-1}\,g(\eta)\;d^\times y\;d^\times x\;d^+\eta$$

The measure $|x| \cdot d^\times x$ is a constant multiple of the *additive* Haar measure $d^+x$. The constant is irrelevant, since it cancels itself in the rearrangement:

$$|x|\,|\eta|^{-1}\; d^\times x\; d^+\eta \;=\; d^+x\; d^\times\eta$$

Continuing,

$$\begin{aligned}
&Z(s,\chi,f)\,Z(1-s,\overline{\chi},\widehat{g})\\
&= \int_{k^\times}\int_k\int_{k^\times} \overline{\psi}(yx)\,f(x)\,\overline{\chi}(y)|y|^{1-s}\,\chi(\eta)|\eta|^s\,g(\eta)\;d^\times y\;d^+x\;d^\times\eta\\
&= \int_{k^\times}\int_{k^\times} \widehat{f}(y)\,\overline{\chi}(y)|y|^{1-s}\,\chi(\eta)|\eta|^s\,g(\eta)\;d^\times y\;d^\times\eta\\
&= Z(1-s,\overline{\chi},\widehat{f})\,Z(s,\chi,g)
\end{aligned}$$

This proves the local functional equation in $0 < \mathrm{Re}(s) < 1$. The general assertion follows from the Identity Principle. ///

**Complex zeta integrals** [Recap]

The product formula requires

$$|z|_{\mathbb{C}} \;=\; |N^{\mathbb{C}}_{\mathbb{R}}(z)|_{\mathbb{R}} \;=\; |z|^2 \qquad \text{(the latter the \textit{usual} norm)}$$

Similarly, the *additive character* $\psi_{\mathbb{C}}(z)$ is

$$\psi_{\mathbb{C}}(z) \;=\; \psi_{\mathbb{R}}(\mathrm{tr}^{\mathbb{C}}_{\mathbb{R}}(z)) \;=\; e^{2\pi i(z+\overline{z})} \;=\; e^{4\pi i \mathrm{Re}\,(z)}$$

The proper normalization of measure on $\mathbb{C}$ for Iwasawa-Tate is

$$d_{\mathbb{C}}(z) \;=\; 2 \cdot d_{\mathrm{usual}}(z)$$

The proper normalization of Gaussian to be its own Fourier transform is

$$f_{\mathbb{C}}(w) \;=\; e^{-2\pi w\overline{w}}$$

The standard *unramified* complex zeta integral is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s \, e^{-2\pi z\overline{z}} \, \frac{2\,dz}{|z|_{\mathbb{C}}} \;=\; 2\pi \cdot (2\pi)^{-s}\, \Gamma(s)$$

The *ramified* unitary characters of $\mathbb{C}^\times$ are

$$\chi_\ell(re^{i\theta})|re^{i\theta}|_{\mathbb{C}}^s \;=\; e^{i\ell\theta} \cdot r^{2s} \qquad\qquad (\text{for } \ell \in \mathbb{Z})$$

The standard choice of Schwartz function for the complex zeta integral depends on the *sign* of $\ell \in \mathbb{Z}$. For $\ell \geq 0$, it is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s \;\; \chi_\ell(z) \;\; \overline{z}^\ell \;\; e^{-2\pi z\overline{z}} \, \frac{2\,dz}{|z|_{\mathbb{C}}} \;=\; 2\pi \cdot (2\pi)^{-(s+\frac{\ell}{2})}\, \Gamma\big(s + \frac{\ell}{2}\big)$$

For $\ell \le 0$, the standard ramified complex local zeta integral is

$$\int_{\mathbb{C}} |z|_{\mathbb{C}}^s \;\; \chi_\ell(z) \;\; z^{|\ell|} \;\; e^{-2\pi z\overline{z}} \; \frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$= \int_0^{2\pi} \int_0^\infty r^{2s} \; e^{i\ell\theta} \; (re^{i\theta})^{-\ell} \; e^{-2\pi z\overline{z}} \; \frac{2\,dz}{|z|_{\mathbb{C}}}$$

$$= \; 4\pi \int_0^\infty r^{2s-\ell} \; e^{-2\pi r^2} \; r \, \frac{dr}{r^2} \; = \; 4\pi \int_0^\infty r^{2s-\ell} \; e^{-2\pi r^2} \; \frac{dr}{r}$$

$$= \; 2\pi \int_0^\infty r^{s-\frac{\ell}{2}} \; e^{-2\pi r} \; \frac{dr}{r} \; = \; 2\pi \cdot (2\pi)^{-(s-\frac{\ell}{2})} \int_0^\infty r^{s-\frac{\ell}{2}} \; e^{-r} \; \frac{dr}{r}$$

$$= \; 2\pi \cdot (2\pi)^{-(s-\frac{\ell}{2})} \, \Gamma\big(s - \frac{\ell}{2}\big) \qquad\qquad (\text{for } \ell \le 0)$$

Thus, for *both* $\ell \ge 0$ and $\ell \le 0$, the local zeta integral is

$$2\pi \cdot (2\pi)^{-(s+\frac{|\ell|}{2})} \, \Gamma\big(s + \frac{|\ell|}{2}\big) \qquad\qquad (\text{for both } \ell \ge 0 \text{ and } \ell \le 0)$$

**Convergence of local zeta integrals** in $\mathrm{Re}\, s > 0$:

As usual, suppose $\chi_v$ is *unitary* meaning $|\chi_v| = 1$, since any non-unitary part could be absorbed into $|\cdot|^s$. Use the standard notation $\sigma = \mathrm{Re}(s)$.

Treat the non-archimedean case first. Since $f_v \in \mathscr{S}(k_v)$, for some $n$ the support of $f$ is contained in $\varpi^{-n}\mathfrak{o}_v$. Since $f$ is locally constant and compactly supported, it has a finite bound $C$. Then

$$|Z_v(s,\chi_v,f_v)| \;\le\; \int_{k_v^\times} |x|^\sigma\,|\chi(x)|\,|f(x)|\,dx$$

$$\le\; C\cdot \int_{k_v^\times \cap \varpi^{-n}\mathfrak{o}_v} |x|^\sigma\,dx \;=\; C\cdot \int_{(k_v^\times \cap \varpi^{-n}\mathfrak{o}_v)/\mathfrak{o}_v^\times} |x|^\sigma \Big(\int_{\mathfrak{o}_v^\times} 1\Big)\,dx$$

$$=\; C\cdot \sum_{\ell=-n}^{\infty} |\varpi^\ell|_v^\sigma \;=\; C\cdot \frac{q_v^{n\sigma}}{1-q_v^{-\sigma}} \;<\; \infty \qquad\qquad (\text{for } \sigma = \mathrm{Re}(s) > 0)$$

For $k_v \approx \mathbb{R}$, given $f \in \mathscr{S}(\mathbb{R})$ for each $N$ that

$$|f(x)| \ll_N (1+|x|^2)^{-N}$$

With $\sigma = \mathrm{Re}(s) > 0$, the local zeta integral is

$$|Z_v(s, \chi_v, f_v)| \ll_N \int_{\mathbb{R}^\times} |x|^\sigma \, |\chi_v(x)| \, (1+x^2)^{-N} \, \frac{dx}{|x|}$$

$$\ll \int_0^\infty |x|^{\sigma-1} \, (1+x^2)^{-N} \, dx$$

$$\ll \int_0^1 |x|^{\sigma-1} \, dx \; + \; \int_1^\infty |x|^{\sigma-1-2N} \, dx$$

Given $\sigma > 0$, take $N$ large enough so that $\sigma - 1 - 2N < -1$ gives convergence. ///

Convergence of the complex integrals is similar...

**Archimedean Fourier transform eigenfunctions (Hecke's identity)**

Whenever possible, we want local Schwartz functions $f_v$ which are eigenfunctions for Fourier transform, preferably unchanged.

We want this so that in the *global* functional equation $Z(s,\chi,f) = Z(1-s,\overline{\chi},\widehat{f})$ as many local factors as possible are the same on both sides, apart from $s \leftrightarrow 1-s$ and $\chi \leftrightarrow \overline{\chi}$.

For absolutely unramified $k_v/\mathbb{Q}_p$, the characteristic function of the local integers $\mathfrak{o}_v$ is its own Fourier transform. For $\chi_v$ unramified at $v$, this gives the desired symmetry.

On $\mathbb{R}$, the Gaussian $e^{-\pi x^2}$ is its own Fourier transform.

On $\mathbb{R}$, for ramified $\chi_{\mathbb{R}}$, that is, for $\mathrm{sgn}(x)\,|x|^s$, the function $f_v(x) = xe^{-\pi x^2}$ is multiplied by $-i$ under Fourier transform.

*Proof:* One argument is by contour-shifting:

$$\widehat{f_v}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\; x\, e^{-\pi x^2}\; dx$$

$$=\; \int_{\mathbb{R}} e^{-2\pi i\xi(x-i\xi)}\; (x-i\xi)\, e^{-\pi(x-i\xi)^2}\; dx$$

$$=\; e^{-\pi\xi^2} \int_{\mathbb{R}} (x-i\xi)\, e^{-\pi x^2}\; dx \;=\; e^{-\pi\xi^2}\cdot\big(0-i\xi\big) \;=\; -i\xi e^{-\pi\xi^2}$$

Done. ///

**Remark:** While contour-shifting arguments have certain virtues, their computation-intense, somewhat non-conceptual nature is not as helpful as we might hope.

Fourier transforms of Schwartz functions for ramified characters on $k_v \approx \mathbb{C}$ are the critical sub-case of *Hecke's identity* on $\mathbb{R}^n$. With, *usual* normalizations:

**Claim:** The Schwartz function $(x \pm iy)^\ell \, e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-\ell}$.

*Proof:* Just do the case $(x + iy)^\ell \, e^{-\pi(x^2+y^2)}$. Rewrite this as $z^\ell e^{-\pi z\overline{z}}$, and rewrite the Fourier transform as

$$\int_{\mathbb{C}} e^{-\pi i(z\overline{w}+\overline{z}w)} \, z^\ell \, e^{-\pi z\overline{z}} \, dz$$

$$= (-\pi i)^{-\ell}\Big(\frac{\partial}{\partial\overline{w}}\Big)^\ell \int_{\mathbb{C}} e^{-\pi i(z\overline{w}+\overline{z}w)} \, e^{-\pi z\overline{z}} \, dz \;=\; (-\pi i)^{-\ell}\Big(\frac{\partial}{\partial\overline{w}}\Big)^\ell e^{-\pi w\overline{w}}$$

$$= (-\pi i)^{-\ell}(-\pi w)^\ell \, e^{-\pi w\overline{w}} \;=\; i^{-\ell} \cdot w^\ell \, e^{-\pi w\overline{w}}$$

This presumes $\partial/\partial\overline{w}$ works as expected, which it does. ///

**Hecke's identity:** Let $P$ be a homogeneous, degree $d$ *harmonic* polynomial on $\mathbb{R}^n$, meaning that $\Delta P = 0$, where $\Delta = \sum_j \partial^2/\partial x_j^2$ is the usual Laplacian. Let $\langle x, \xi \rangle = \sum_j x_j \xi_j$ be the usual pairing. Then $P(x)\, e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue $i^{-d}$:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; i^{-d} \cdot P(\xi)\, e^{-\pi|\xi|^2}$$

[*Proof later.*]

**Remark:** Any other non-degenerate pairing $x \times \xi \to \langle x, \xi \rangle$ and suitable associated operator $\Delta$ works the same way.

**Remark:** Since $f^{\frown\frown}(x) = f(-x)$, necessarily $f^{\frown\frown\frown\frown} = f$, for all $f$. Thus, the only possible eigenvalues of Fourier transform are $\pm 1, \pm i$. Further, the corresponding components are easy to pick out: with $\varepsilon \in \{\pm 1, \pm i\}$, the $\varepsilon^{th}$ component of $f$ is

$$f \;+\; \varepsilon^{-1}\widehat{f} \;+\; \varepsilon^{-2} f^{\frown\frown} \;+\; \varepsilon^{-3} f^{\frown\frown\frown}$$