• *Recap:* **Convergence of half-zeta integrals** Genuinely prove convergence of the half-zeta integrals

$$\int_{\mathbb{J}^+} |y|^s\, f(y)\, dy \;=\; \int_{\mathbb{J}^+/k^\times} |y|^s\, \theta_f^*(y)\, dy$$

with f a Schwartz function on the adeles, for *all* $s \in \mathbb{C}$, where $\theta_f^*(y) = \sum_{\alpha\in k^\times} f(\alpha y)$.

• **Interlude:** Harmonic analysis on spheres, representation theory of orthogonal groups $O(n,\mathbb{R})$ or $SO(n,\mathbb{R})$, to prove

Hecke's identity: For a homogeneous, degree d *harmonic* polynomial P on $\mathbb{R}^n$, $P(x)\, e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue i^{-d}:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)\widehat{\;}(\xi) \;=\; i^{-d}\cdot P(\xi)\, e^{-\pi|\xi|^2}$$

Remark: The proof of Hecke's identity illustrates the power of *representation theory.*

Proof: Whether or not P is harmonic,

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi,x\rangle}\; P(x)\, e^{-\pi|x|^2}\; dx$$

$$=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial \xi}\Big) \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi,x\rangle}\; e^{-\pi|x|^2}\; dx$$

because

$$P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial \xi_1},\ldots,\frac{1}{-2\pi i}\frac{\partial}{\partial \xi_n}\Big) e^{-2\pi i\langle \xi,x\rangle} \;=\; P(x)$$

Since the Gaussian is its own Fourier transform,

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\, e^{-\pi|\xi|^2}$$

whether or not P is harmonic. Certainly

$$P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\, e^{-\pi|\xi|^2} \;=\; P^{\#}(\xi)\, e^{-\pi|\xi|^2}$$

for a polynomial $P^{\#}$ of total degree at most that of P. Since Fourier transform commutes with the action of $O(n,\mathbb{R})$ on functions,

$$\Big((P\circ g)(x)\, e^{-\pi|x|^2}\Big)^\frown(\xi) \;=\; \Big(P(gx)\, e^{-\pi|gx|^2}\Big)^\frown(\xi)$$

$$=\; \Big(P(x)\, e^{-\pi|x|^2}\Big)^\frown(g\xi) \;=\; P^{\#}(g\xi)\, e^{-\pi|\xi|^2}$$

Thus, $P \to P^{\#}$ is an $O(n,\mathbb{R})$-map:

$$(P\circ g)^{\#} \;=\; P^{\#}\circ g \qquad\qquad (\text{for } g\in O(n,\mathbb{R}))$$

Thus, $P \to P^{\#}$ gives an $O(n,\mathbb{R})$-respecting map of the space V_d, of *all* polynomials of total degree at most d, to itself.

The sequel: we will show... first, the space H_d of homogeneous degree-d harmonic polynomials is *irreducible* as $O(n,\mathbb{R})$-representation, meaning that it has no proper vector subspace stable under $O(n,\mathbb{R})$.

Second, as $O(n,\mathbb{R})$-representation space, meaning as complex vector space with linear action of $O(n,\mathbb{R})$,

$$V_d \;=\; H_d \,\oplus\, \bigoplus (\text{other irreducibles } \pi \not\approx H_d)$$

Third, *any* $O(n,\mathbb{R})$-respecting map $V_d \to V_d$ maps H_d to itself.

Fourth, (an instance of *Schur's Lemma*) that any $O(n,\mathbb{R})$-map of *any* irreducible to itself is a *scalar*.

Fifth, the two-variable case determines the constant i^{-d}.

- Required properties
- Existence of the spherical Laplacian
- Polynomial eigenvectors for the spherical Laplacian
- Determination of eigenvectors
- Existence of invariant integrals on spheres
- L^2 spectral decompositions on spheres

We will see that spheres $S^{n-1} \subset \mathbb{R}^n$, are quotients

$$S^{n-1} \;\approx\; SO(n-1) \setminus SO(n)$$

of *rotation groups* (orthogonal groups) $SO(n)$. Spheres themselves are rarely groups, but *are* acted-upon transitively by groups.

It is well known that S^1 is a group, and also

$$S^3 \;\approx\; \{\text{quaternions } a+bi+ch+dk \;:\; a^2+b^2+c^2+d^2=1\}$$

Write Δ^S for the desired rotation-invariant second-order differential operator (Laplacian) on functions on $S = S^{n-1}$, and $\int_S f$ the desired rotation-invariant integral. Two characterizing properties are

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_S f \cdot (\Delta^S \varphi) \quad (\textbf{self-adjointness})$$

$$\int_S (\Delta^S f) \cdot \overline{f} \;\leq\; 0 \quad (\textbf{definiteness})$$

with equality only for f *constant*. Assume also that Δ^S has *real coefficients*, in the sense that $\overline{\Delta^S f} = \Delta^S\,\overline{f}$.

There is the natural complex hermitian inner product

$$\langle f, g \rangle \;=\; \int_S f \cdot \overline{g} \qquad \text{(for differentiable functions } f, g \text{ on } S\text{)}$$

A typical linear algebra conclusion, via a typical argument:

Corollary: Granting Δ^S and invariant measure on S^{n-1}... , eigenvectors f, g for Δ^S with *distinct* eigenvalues are *orthogonal* with respect to $\langle , \rangle$. Eigenvalues are *non-positive real* numbers.

Proof: Let $\Delta^S f = \lambda \cdot f$ and $\Delta^S g = \mu \cdot g$. Assume $\lambda \neq 0$ (or else interchange the roles of λ and μ). Then

$$\langle f, f \rangle \;=\; \frac{1}{\lambda} \int_S (\Delta^S f) \cdot \overline{f} \;=\; \frac{1}{\lambda} \int_S f \, \overline{\Delta^S f} \;=\; \frac{\overline{\lambda}}{\lambda} \int_S f \, \overline{f}$$

Since $\lambda \neq 0$, f is not identically 0, so the integral of $f \cdot \overline{f}$ is not 0, and $\lambda = \overline{\lambda}$, so λ is *real*. The negative definiteness of Δ^S and positive-ness of the invariant measure on S give

$$\lambda \cdot \langle f, f \rangle \;=\; \int_S (\Delta^S f) \cdot \overline{f} \;<\; 0$$

Next,

$$\langle f, g\rangle \;=\; \frac{1}{\lambda}\int_S (\Delta^S f)\cdot \overline{g} \;=\; \frac{1}{\lambda}\int_S f\cdot \overline{\Delta^S g} \;=\; \frac{\overline{\mu}}{\lambda}\int_S f\cdot \overline{g}$$

The eigenvalues λ, μ are real, so for $\mu/\lambda \neq 1$ necessarily the integral is 0. ///

The standard **special orthogonal group** (=rotation group)

$$SO(n) \;=\; \{g \in GL_n(\mathbb{R}) \,:\, g^\top g = 1_n \quad \text{and} \quad \det g = 1\}$$

acts on S by *right* matrix multiplication,

$$k \times x \longrightarrow xk \qquad\qquad (\text{for } x \in S^{n-1} \text{ and } k \in O(n))$$

considering elements of $\mathbb{R}^n$ as *row* vectors.

Claim: The action of $SO(n)$ on S^{n-1} is *transitive*. ///

The *isotropy group* $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\text{(isotropy group)} \;=\; SO(n)_{e_n} \;=\; \{\begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SO(n-1)\}$$

$$\approx\; SO(n-1)$$

Thus, by transitivity, as $SO(n)$-spaces $S^{n-1} \;\approx\; SO(n-1)\backslash SO(n)$

The action of $k \in SO(n)$ on *functions* f on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) is $(k \cdot f)(x) \;=\; f(xk)$. The rotation invariance conditions are

$$\int_S k \cdot f \;=\; \int_S f \qquad \text{(for } k \in SO(n))$$

$$\Delta^S(k \cdot f) \;=\; k \cdot (\Delta^S f) \quad \text{(for } k \in SO(n))$$

The spherical Laplacian Grant that the usual Euclidean Laplacian

$$\Delta = \left(\frac{\partial}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial}{\partial x_n}\right)^2$$

is $SO(n)$-invariant. For f on S, create a function F on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f = (\text{restriction to } S \text{ of }) \, \Delta F$$

The map $f \to F$ that creates from f on S the degree-zero positive-homogeneous function F on $\mathbb{R}^n - 0$ *commutes with* the action of $SO(n)$. From the definition,

$$\Delta^S \overline{f} = \overline{\Delta^S f}$$

The $SO(n)$-invariance of the spherical Laplacian follows from the $SO(n)$-invariance of the usual Laplacian: for $k \in SO(n)$

$$\Delta^S(k \cdot f) \;=\; (\Delta(k \cdot F))|_S \;=\; (k \cdot (\Delta F))|_S \;=\; k \cdot (\Delta F)|_S$$

since restriction to the sphere commutes with $SO(n)$, as does $f \to F$. Thus, Δ^S is $SO(n)$-invariant.

Claim: For f positive-homogeneous of degree s on $\mathbb{R}^n - 0$

$$\Delta(|x|^{-s}\, f) \;=\; -s(s+n-2)|x|^{-(s+2)}\, f + |x|^{-s}\, \Delta f$$

Corollary: For f positive-homogeneous of degree s and *harmonic*, the restriction $f|_S$ of f to S^{n-1} is an *eigenfunction* for Δ^S,

$$\Delta^S(f|_S) \;=\; -s(s+n-2) \cdot (f|_S)$$

Proof: *(of claim)* Computing directly, with $r = |x|$ and f_i be the partial derivative with respect to the i^{th} argument,

$$\Delta^S(f|_S) \;=\; \Delta f(x/|x|) \;=\; \Delta\left(|x|^{-s}\cdot f\right) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2}\left((r^2)^{-\frac{s}{2}}\cdot f\right)$$

$$=\; \sum_i \frac{\partial}{\partial x_i}\Big(-\frac{s}{2}\,(2x_i)\,(r^2)^{-(\frac{s}{2}+1)}\,f + (r^2)^{-s/2}\,f_i Big))$$

$$=\; \sum_i \frac{\partial}{\partial x_i}\left(-sx_i\,(r^2)^{-(\frac{s}{2}+1)}\,f + (r^2)^{-s/2}\,f_i\right)$$

$$=\; \sum_i \Big(-s\,(r^2)^{-(\frac{s}{2}+1)}\,f + sx_i\,(\frac{s}{2}+1)(2x_i)\,(r^2)^{-(\frac{s}{2}+2)}\,f$$

$$-sx_i(r^2)^{-(\frac{s}{2}+1)}\,f_i \;-\; \frac{s}{2}\,(2x_i)(r^2)^{-(\frac{s}{2}+1)}\,f_i \;+\; (r^2)^{-s/2}\,f_{ii}\Big)$$

which simplifies to

$$-ns(r^2)^{-(\frac{s}{2}+1)}f + sr^2(s+2)\,(r^2)^{-(\frac{s}{2}+2)}f$$

$$-s(r^2)^{-(\frac{s}{2}+1)}sf + (r^2)^{-s/2}\Delta f$$

using $\sum_i x_i^2 = r^2$ and *Euler's identity:* for positive-homogeneous f of degree s,

$$\sum_i x_i\; f_i(x) \;=\; s\cdot f$$

Euler's identity is proven by considering the function $g(t) = f(tx)$ for $t > 0$, differentiating with respect to t, and evaluating at $t = 1$.

Simplifying,

$$\Delta(|x|^{-s}f) = -ns\,r^{-(s+2)}f + s(s+2)\,r^{-(s+2)}f - 2s\,r^{-(s+2)}\,sf + r^{-s}\Delta f$$

$$= \;-s(n-(s+2)+2s)\,r^{-(s+2)}\,f + r^{-s}\,\Delta f$$

$$= \;-s(n+s-2)\,r^{-(s+2)}\,f \;+\; r^{-s}\,\Delta f \qquad \text{as asserted.} \;\;///$$

Remark: The most tractable homogeneous functions are homogeneous *polynomials*, so we look for *harmonic* homogeneous polynomials before anything subtler.

Gratifyingly, a slightly more sophisticated argument proves that there are no *other* eigenfunctions of the spherical Laplacian.

Let H_d be homogeneous (total) degree d harmonic elements in $\mathbb{C}[x_1, \ldots, x_n]$, and $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ the *homogeneous* polynomials of degree d. Introduce a complex-hermitian form

$$(,) \; : \; \mathbb{C}[x_1, \ldots, x_n] \times \mathbb{C}[x_1, \ldots, x_n] \longrightarrow \mathbb{C}$$

by

$$(P, Q) \; = \; \overline{Q}(\partial)\,(P(x))\,|_{x=0}$$

where $Q(\partial)$ means to replace x_i by $\partial/\partial x_i$ in a polynomial, and $R|_{x=0}$ means to evaluate R at $x = 0$.

Multiplication by r^2 is *adjoint* to application of Δ:

$$(\Delta f, g) \;=\; (f, r^2 g) \qquad\qquad (\text{with } r^2 = x_1^2 + \ldots + x_n^2)$$

Claim: The pairing $(,)$ is positive-definite hermitian.

Proof: For homogeneous polynomials, $(P, Q) = 0$ unless P, Q are of the same degree. When restricted to $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$, the form $(,)$ has an *orthogonal basis* of *distinct monomials*, since

$$\left(\frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdots \frac{\partial^{m_n}}{\partial x_n^{m_n}}\right)(x_1^{e_1} \ldots x_n^{e_n})\Big|_{x=0}$$

$$= \begin{cases} 0 & (\text{if } \textit{any } m_i \neq e_i) \\ m_1! \ldots m_n! & (\text{if } \textit{every } m_i = e_i) \end{cases} \qquad ///$$

Looking at the orthogonal basis of monomials, $(,)$ is *hermitian* and *positive definite* on $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$. ///