• **Interlude:** Calculus on spheres: invariant integrals, invariant $\Delta = \Delta^S$, integration-by-parts, etc.

Decomposition of $L^2(S^{n-1})$ into Δ^S-eigenfunctions.

Representation theory of orthogonal groups $O(n,\mathbb{R})$ or $SO(n,\mathbb{R})$.

... combine to prove

Hecke's identity: For a homogeneous, degree d *harmonic* polynomial P on $\mathbb{R}^n$, $P(x)\, e^{-\pi|x|^2}$ is a Fourier transform eigenfunction with eigenvalue i^{-d}:

$$\Big(P(x)\, e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; i^{-d}\cdot P(\xi)\, e^{-\pi|\xi|^2}$$

Proof recap: Whether or not P is harmonic,

$$\Big(P(x)\,e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\langle \xi,x\rangle}\; P(x)\,e^{-\pi|x|^2}\;dx$$

$$=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\int_{\mathbb{R}^n} e^{-2\pi i\langle \xi,x\rangle}\; e^{-\pi|x|^2}\;dx$$

$$=\; P\Big(\frac{1}{-2\pi i}\frac{\partial}{\partial\xi}\Big)\,e^{-\pi|\xi|^2} \;=\; P^{\#}(\xi)\,e^{-\pi|\xi|^2}$$

for a polynomial $P^{\#}$ of total degree at most that of P. Since Fourier transform commutes with the action of $O(n,\mathbb{R})$ on functions,

$$\Big((P\circ g)(x)\,e^{-\pi|x|^2}\Big)\widehat{\ }(\xi) \;=\; P^{\#}(g\xi)\,e^{-\pi|\xi|^2}$$

Thus, $P \;\to\; P^{\#}$ is an $O(n,\mathbb{R})$-map: $(P\circ g)^{\#} \;=\; P^{\#}\circ g$ for $g\in O(n,\mathbb{R})$.

Write Δ^S for a/the rotation-invariant second-order differential operator (Laplacian) on functions on $S = S^{n-1}$, and $\int_S f$ the rotation-invariant integral. Two characterizing properties are

$$\int_S (\Delta^S f)\cdot\varphi \;=\; \int_S f\cdot(\Delta^S\varphi) \quad (\textbf{self-adjointness})$$

$$\int_S (\Delta^S f)\cdot\overline{f} \;\le\; 0 \quad (\textbf{definiteness})$$

with equality only for f *constant*. Assume also that Δ^S has *real coefficients*, in the sense that $\overline{\Delta^S f} = \Delta^S\,\overline{f}$.

There is the natural complex hermitian inner product

$$\langle f, g\rangle \;=\; \int_S f\cdot\overline{g} \qquad \text{(for differentiable functions } f, g \text{ on } S\text{)}$$

Corollary: Δ^S-eigenvectors f, g with *distinct* eigenvalues are *orthogonal.* Eigenvalues are *non-positive real.* ///

Claim: The action of $SO(n)$ on S^{n-1} is *transitive.* ///

The *isotropy group* $SO(n)_{e_n}$ of the last standard basis vector $e_n = (0, \ldots, 0, 1)$ is

$$\{ \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} : A \in SO(n-1)\} \;\approx\; SO(n-1)$$

By transitivity, as $SO(n)$-spaces $S^{n-1} \;\approx\; SO(n-1)\backslash SO(n)$

The action of $k \in SO(n)$ on *functions* f on the sphere $S = S^{n-1}$ (or on the ambient $\mathbb{R}^n$) is $(k \cdot f)(x) \;=\; f(xk)$. The rotation invariance conditions are

$$\int_S k \cdot f = \int_S f \qquad \Delta^S(k \cdot f) = k \cdot (\Delta^S f) \qquad (\text{for } k \in SO(n))$$

The spherical Laplacian For f on S, create a function F on $\mathbb{R}^n - 0$ by $F(x) = f(x/|x|)$, and define

$$\Delta^S f \;=\; (\Delta F)\big|_S$$

Then $\Delta^S \overline{f} = \overline{\Delta^S f}$ and Δ^S is $SO(n)$-invariant.

Claim: For f positive-homogeneous of degree s on $\mathbb{R}^n - 0$

$$\Delta(|x|^{-s} f) \;=\; -s(s+n-2)|x|^{-(s+2)} f + |x|^{-s} \Delta f$$

Corollary: For f positive-homogeneous of degree s and *harmonic*, the restriction $f|_S$ of f to S^{n-1} is an *eigenfunction* for Δ^S,

$$\Delta^S(f|_S) = -s(s+n-2) \cdot (f|_S)$$

The proof is a direct computation, except for one interesting fact, *Euler's identity:*

$$\sum_i x_i\, f_i(x) \;=\; s \cdot f \qquad\qquad (f \text{ positive-homogeneous degree } s)$$

Euler's identity is proven by considering the function $g(t) = f(tx)$ for $t > 0$, differentiating with respect to t, and evaluating at $t = 1$.

Define complex-hermitian $(,)$ on $\mathbb{C}[x_1, \ldots, x_n]$ by

$$(P, Q) \;=\; \overline{Q}(\partial)\,(P(x))\,|_{x=0}$$

where $Q(\partial)$ means to replace x_i by $\partial/\partial x_i$ in a polynomial, and $R|_{x=0}$ means to evaluate R at $x = 0$.

Multiplication by r^2 is *adjoint* to application of Δ:

$$(\Delta f, g) \;=\; (f, r^2 g) \qquad\qquad (\text{with } r^2 = x_1^2 + \ldots + x_n^2)$$

Claim: $\Delta \;:\; \mathbb{C}[x_1, \ldots, x_n]^{(d)} \;\longrightarrow\; \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$
is *surjective.* Harmonic polynomials f in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$ are *orthogonal* to polynomials $r^2 h$ with $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$.

Proof: For $h \in \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$, if $(\Delta f, h) = 0$ for all f in $\mathbb{C}[x_1, \ldots, x_n]^{(d)}$, then

$$0 \;=\; (\Delta f, h) \;=\; (f, r^2 h) \qquad\qquad (\text{for all } f)$$

so $r^2 h = 0$, so $h = 0$, by the positive-definiteness of $(,)$. This also proves the second assertion. ///

Corollary: $\mathbb{C}[x_1, \ldots, x_n]^{(d)} \;=\; H_d \oplus r^2 H_{d-2} \oplus r^4 H_{d-4} + \ldots$ ///

Corollary: Polynomials restricted to the n-sphere are equal to linear combinations of *harmonic* polynomials.

Proof: Use the observation

$$\mathbb{C}[x_1, \ldots, x_n]^{(d)} = H_d \oplus r^2 H_{d-2} \oplus r^4 H_{d-4} + \ldots$$

to write a homogeneous polynomial as

$$f = f_0 + r^2 f_2 + r^4 f_4 + \ldots$$

with each f_i harmonic. Restricting to the sphere,

$$f|_S = (f_0 + r^2 f_2 + r^4 f_4 + \ldots)|_S = (f_0 + f_2 + f_4 + \ldots)|_S$$

since $r^2 = 1$ on the sphere. ///

Remark: From computations above,

$$\Delta^S f \;=\; -d(d+n-2)\cdot f \qquad\qquad (\text{for } f \in H_d)$$

Since $d \geq 0$,

$$\lambda_d \;=\; -d(d+n-2) = -\Big(d + \frac{n-2}{2}\Big)^2 + \Big(\frac{n-2}{2}\Big)^2 \;\leq\; 0$$

The eigenvalues $\lambda_d = -d(d+n-2)$ are *strictly decreasing* as $d \to +\infty$, so the spaces H_d are *distinguished* by their eigenvalues for the spherical Laplacian.

Remark: For S^1, the 0-eigenspace is 1-dimensional and for $d > 0$ the $(-d^2)$-eigenspace is 2-dimensional, with basis $(x \pm iy)^d$. In contrast, for $n > 1$ the dimensions of eigenspaces are *unbounded* as the degree d goes to $+\infty$. Specifically, ...

Claim: $\dim_{\mathbb{C}} H_d \;=\; \dim \mathbb{C}[x_1, \ldots, x_n]^{(d)} - \dim \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$

$$= \binom{n+d-1}{n-1} - \binom{n+d-3}{n-1} \sim \text{constant} \cdot d^{n-2}$$

Proof: From above, $\Delta : \mathbb{C}[x_1, \ldots, x_n]^{(d)} \longrightarrow \mathbb{C}[x_1, \ldots, x_n]^{(d-2)}$ is surjective, so $\dim H_d$ is the difference of dimensions.

The dimension of total-degree d polynomials in n variables is the number of monomials $x_1^{e_1} \ldots x_n^{e_n}$ with $\sum_i e_i = d$. Imagine each exponent as the corresponding number of marks, with $n-1$ *additional* marks to separate the marks corresponding to the n distinct variables x_i, for a total of $n+d-1$. The choice of location of the separating marks is the binomial coefficient. ///

Corollary (instance of *Weyl's Law)* The dimension of the direct sum of (polynomial) Δ^S-eigenspaces with $|\lambda| < T$ grows like $T^{\frac{n-1}{2}} = T^{\frac{1}{2}\dim S^{n-1}}$. ///

Invariant integrals on spheres, integration by parts for Δ^S.

We have used an $SO(n)$-invariant integral on S^{n-1} to show that eigenvalues for the spherical Laplacian Δ^S are non-positive, in determining all eigenvectors, using *integration by parts* on S^{n-1}.

Instead of invoking Haar measure, we could write a *formula* as follows, using $SO(n)$-invariance of the measure on $\mathbb{R}^n$. For continuous f on S, define

$$\int_S f \;=\; \int_{\mathbb{R}^n - 0} \gamma(|x|^2)\; f(x/|x|)\; dx$$

where γ is a fixed smooth non-negative function on $[0, \infty)$ with

$$\int_{\mathbb{R}^n} \gamma(|x|^2)\, dx \;=\; 1$$

For convenience, we may at some moments suppose that γ has compact support and vanishes identically on a neighborhood of 0.

For $k \in SO(n)$ we have the $SO(n)$-invariance of this integral:

$$\int_S k \cdot f \;=\; \int_{\mathbb{R}^n - 0} \gamma(|x|^2)\, f(\frac{xk}{|xk|})\, dx \;=\; \int_{\mathbb{R}^n - 0} \gamma(|xk^{-1}|^2)\, f(\frac{x}{|x|})\, dx$$

$$= \int_{\mathbb{R}^n - 0} \gamma(|x|^2)\, f(\frac{x}{|x|})\, dx \;=\; \int_S f$$

by changing variables to replace x by xk^{-1}, and using $|xk^{-1}| = |x|$. Less trivial is proof of the desired integration-by-parts-twice result from this clunky viewpoint:

Proposition: For differentiable functions f, φ on S^n,

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_S f \cdot \Delta^S \varphi$$

Further, Δ^S is *negative-definite* in the sense that $\int_S (\Delta^S f) \cdot \overline{f} \le 0$ with equality only for f *constant*.

Proof: Let $F(x) = f(x/r)$ and $\Phi(x) = \varphi(x/r)$. By definition,

$$\int_S (\Delta^S f) \cdot \varphi \;=\; \int_{\mathbb{R}^n - 0} \gamma(r^2)\, r^2 \cdot (\Delta F)(x)\; \Phi(x)\; dx$$

where r^2 is inserted so $r^2 \Delta F$ is positive-homogeneous of degree 0 as required. Integrating by parts on $\mathbb{R}^n$, this becomes

$$-\int_{\mathbb{R}^n - 0} \sum_i \frac{\partial F}{\partial x_i}\; \frac{\partial}{\partial x_i} \big(r^2 \cdot \gamma(r^2)\; \Phi(x) \big)\; dx$$

With $\beta(r^2) = r^2 \gamma(r^2)$, the derivative $\frac{\partial}{\partial x_i} \big[r^2 \cdot \gamma(r^2)\, \Phi(x) \big]$ is

$$\frac{\partial}{\partial x_i} \big[\beta(r^2)\; \Phi(x) \big] \;=\; 2x_i \beta'(r^2) \Phi(x) + \beta(r^2) \frac{\partial \Phi}{\partial x_i}$$

Thus, the whole is

$$-\int_{\mathbb{R}^n-0}\sum_i \frac{\partial F}{\partial x_i}\ \left[2x_i\beta'(r^2)\Phi(x)+\beta(r^2)\frac{\partial \Phi}{\partial x_i}\right]\ dx$$

$$=\ -\int_{\mathbb{R}^n-0}\sum_i \frac{\partial F}{\partial x_i}\ \beta(r^2)\frac{\partial \Phi}{\partial x_i}\ dx$$

since by Euler's identity $\sum_i x_i \frac{\partial F}{\partial x_i} = (\text{degree } F)\cdot F = 0$. The last expression for the integral is symmetric in F and Φ. And with $\Phi = \overline{F}$ the last expression is non-positive, and 0 only for $\partial F/\partial x_i = 0$ for all i, only if F is constant, only if f is constant. ///

Remark: A more persuasive argument will be given later.

Spectral decomposition of $L^2(S^{n-1})$ Functions on the sphere should be *sums of eigenfunctions* for Δ^S, with convergence in L^2. L^2 convergence does *not* imply pointwise convergence, but for *smooth* functions eventually prove convergence in $C^\infty(S^{n-1})$.

Theorem:

$$L^2(S^{n-1}) \;=\; \text{completion} \bigoplus_{d\geq 0} H_d\Big|_{S^{n-1}} \qquad \text{(orthogonal direct sum)}$$

Proof: For *completeness*, we will prove that restrictions to the sphere of harmonic polyomials are dense in $C^o(S^{n-1})$, which is dense in $L^2(S^{n-1})$.

With $S^{n-1} \subset \mathbb{R}^n$, a short-cut is available: invoke Weierstrass approximation to know that polynomials are sup-norm dense in $C^o(E)$ on any compact subset E of $\mathbb{R}^n$. From above, polynomials restricted to S^{n-1} are equal to *harmonic* polynomials. ///

Thus, every L^2 function f on S^{n-1} has an L^2 Fourier-Laplace expansion

$$f \;=\; \sum_{d=0}^{\infty} f_d \qquad\qquad (\text{in } L^2(S^{n-1}))$$

where f_d is the orthogonal projection of f in $L^2(S^{n-1})$ to the space H_d of homogeneous degree d harmonic polynomials restricted to the sphere.

The d^{th} component f_d is an eigenfunction for Δ^S with eigenvalue $\lambda_d = -d(d+n-2)$.

Note: the Δ^S-eigenvalues $\lambda_d = -d(d+n-2)$ on H_d are *distinct*.

Next, we look at this decomposition of $L^2(S^{n-1})$ in terms of the *representation theory* of $SO(n,\mathbb{R})$.