Dedekind zeta functions, class number formulas, ...

$$\zeta_k(s) \;=\; \sum_{0\neq\mathfrak{a}\subset\mathfrak{o}} \frac{1}{N\mathfrak{a}^s} \;=\; \prod_{\mathfrak{p} \text{ prime in } \mathfrak{o}} \frac{1}{1 - N\mathfrak{p}^{-s}}$$

The simplest family of rings of algebraic integers typically *not* PIDs, but with the simple feature of *finitely-many units*, is *complex quadratic* $k = \mathbb{Q}(\sqrt{-D})$ for $D > 0$. Let $h(\mathfrak{o})$ be the *class number*, $\chi(p) = (-D/p)_2$ with conductor N. Then

$$h(\mathfrak{o}) \;=\; \Big| |\mathfrak{o}^\times| \sum_{a \bmod N} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a) \Big|$$

Used: $\chi(p) = (d/p)_2$ is *odd*, meaning $\chi(-1) = -1$, $\Leftrightarrow d < 0$. Absolute value of Gauss sum for χ of conductor N is $\sqrt{N}$. For *odd* χ,

$$L(1,\chi) \;=\; \frac{-\pi i}{\sum_a \overline{\chi}(a) e^{2\pi i a/N}} \sum_{a \bmod N} \chi(a) \cdot \big(\frac{a}{N} - \tfrac{1}{2}\big)$$

Used: For a lattice Λ in $\mathbb{C}$, the Epstein zeta function

$$Z_\Lambda(s) \;=\; \sum_{0\neq\lambda\in\Lambda} \frac{1}{|\lambda|^{2s}}$$

has a meromorphic continuation to $\mathrm{Re}(s) > \frac{1}{2}$ and

$$Z_\Lambda(s) \;=\; \frac{\pi}{\text{co-area}\,\Lambda} \cdot \frac{1}{s-1} \;+\; (\text{holomorphic near } s=1)$$

For complex quadratic k,

$$\zeta_k(s) \;=\; \sum_{[\mathfrak{b}]} \sum_{\mathfrak{a}\sim\mathfrak{b}} \frac{1}{N\mathfrak{a}^s} \;\sim\; \frac{\pi \cdot h(\mathfrak{o})}{|\mathfrak{o}^\times| \cdot \text{coarea}(\mathfrak{o}) \cdot (s-1)} + (\text{holo at } s=1)$$

With $\chi(p) = (-D/p)_2$, from the *factorization* $\zeta_k(s) \;=\; \zeta_(s) \cdot L(s,\chi)$

$$L(1,\chi) \;=\; \frac{\pi \cdot h(\mathfrak{o})}{|\mathfrak{o}^\times| \cdot \text{coarea}(\mathfrak{o})}$$

Example: $D = 3$ gives the Eisenstein integers $\mathfrak{o}$, which we know to have class number 1, since the ring is a PID. Here $|\mathfrak{o}^\times| = 6$.

$$|\mathfrak{o}^\times| \sum_{1 \le a < N/2} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a) \;=\; 6\big(\tfrac{1}{3} - \tfrac{1}{2}\big) \cdot (+1) \;=\; -1$$

Adjust by $\varepsilon = -1$ to obtain $h(\mathfrak{o}) = 1$, indeed.

Example: For $D = 5$, the conductor is $N = 20$ and $|\mathfrak{o}^\times| = 2$.

$$|\mathfrak{o}^\times| \sum_{1 \le a < N/2} \big(\frac{a}{N} - \tfrac{1}{2}\big) \cdot \chi(a)$$

$$= \; 2\Big(\big(\tfrac{1}{20} - \tfrac{1}{2}\big)(+1) + \big(\tfrac{3}{20} - \tfrac{1}{2}\big)\big(\tfrac{-5}{3}\big)_2 + \big(\tfrac{7}{20} - \tfrac{1}{2}\big)\big(\tfrac{-5}{7}\big)_2 + \big(\tfrac{9}{20} - \tfrac{1}{2}\big)\big(\tfrac{-5}{9}\big)_2\Big)$$

$$= \; 2\Big(\tfrac{1}{20} + \tfrac{3}{20} + \tfrac{7}{20} + \tfrac{9}{20} - 2\Big) \;=\; -2$$

So $h(\mathfrak{o}) = 2$. That $h(\mathfrak{o}) > 1$ is not surprising, given

$$2 \cdot 3 \;=\; (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$$

It is non-trivial to give an (non-trivial) *upper* bound on $h(\mathfrak{o})$.

Topologies, completions/limits

An **absolute value** or *norm* $x \to |x|$ on a field k is a non-negative real-valued function on k such that

$$\begin{cases} |x| = 0 \text{ only for } x = 0 & \text{(positivity)} \\ |xy| = |x| \cdot |y| & \text{(multiplicativity)} \\ |x+y| \le |x| + |y| & \text{(triangle inequality)} \end{cases}$$

When $|x+y| \le \max(|x|, |y|)$, the norm is *non-archimedean*, or a *valuation.*

A norm gives k has a metric *topology* by $d(x,y) = |x-y|$. Since $|x| = |x \cdot 1| = |x| \cdot |1|$ we have $|1| = 1$. Also, $|\omega|^n = |\omega^n| = |1|$ for an n^{th} root of unity, so $|\omega| = 1$. Then *reflexivity, symmetry*, and the triangle inequality follow for the metric.

Theorem: Two norms $|*|_1$ and $|*|_2$ on k give the same *non-discrete* topology on a field k if and only if $|*|_1 = |*|_2^t$ for some $0 < t \in \mathbb{R}$.

Proof: If the two norms are related this way, they certainly give the same topology. Conversely, assume they give the same non-discrete topology. Then $|x|_1 < 1$ implies $x^n \to 0$ in the $|*|_1$ topology. Thus, $x^n \to 0$ in the $|*|_2$ topology, so $|x|_2 < 1$. Similarly, if $|x|_1 > 1$, then $|x^{-1}|_1 < 1$, so $|x|_2 > 1$.

Fix y with $|y|_1 > 1$. Given $|x|_1 \geq 1$, there is $t \in \mathbb{R}$ such that $|x|_1 = |y|_1^t$. For rational $a/b > t$, $|x|_1 < |y|_1^{a/b}$, so $|x^b/y^a|_1 < 1$. Then $|x^b/y^a|_2 < 1$, and $|x|_2 < |y|_2^{a/b}$.

Similarly, $|x|_2 > |y|_2^{a/b}$ for $a/b < t$. Thus, $|x|_2 = |y|_2^t$, and

$$|x|_2 \;=\; |y|_2^t \;=\; \Big(|y|_1^{\frac{\log|y|_2}{\log|y|_1}}\Big)^t \;=\; \big(|y|_1^t\big)^{\frac{\log|y|_2}{\log|y|_1}} \;=\; |x|_2^{\frac{\log|y|_2}{\log|y|_1}} \qquad ///$$

The *completion* of k with respect to a metric given by a norm is the usual metric completion, and the norm and metric extend by continuity. Assume k is not *discrete*.

It is reasonable to think of $k = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$ or finite extensions of $\mathbb{Q}_p$, and also $\mathbb{F}_q((x))$ and its finite extensions.

Theorem: Over a complete, non-discrete normed field k,
• A *finite-dimensional* k-vectorspace V has just one Hausdorff topology so that vector addition and scalar multiplication are continuous (a *topological vectorspace* topology). All linear endomorphisms are *continuous*.
• A finite-dimensional k-subspace V of a topological k-vectorspace W is necessarily a *closed* subspace of W.
• A k-linear map $\phi : X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Remark: The main application of this is to finite field extensions V of $k = \mathbb{Q}_p$ or of $k = \mathbb{F}_q((x))$. The argument also succeeds over complete non-discrete *division algebras*.

A subset E of V is **balanced** when $xE \subset E$ for every $x \in k$ with $|x| \le 1$.

Lemma: Let U be a neighborhood of 0 in V. Then U contains a *balanced* neighborhood N of 0.

Proof: By continuity of scalar multiplication, there is $\varepsilon > 0$ and a neighborhood U' of $0 \in V$ so that when $|x| < \varepsilon$ and $v \in U'$ then $xv \in U$. Since k is non-discrete, there is $x_o \in k$ with $0 < |x_o| < \varepsilon$. Since scalar multiplication by a non-zero element is a *homeomorphism*, x_oU' is a neighborhood of 0 and $x_oU' \subset U$. Put

$$N = \bigcup_{|y|\le 1} y(x_oU')$$

$|xy| \le |y| \le 1$ for $|x| \le 1$, so

$$xN = \bigcup_{|y|\le 1} x(yx_oU') \subset \bigcup_{|y|\le 1} yx_oU' = N \qquad ///$$

Proposition: For a one-dimensional topological vectorspace V, that is, a free module on one generator e, the map $k \to V$ by $x \to xe$ is a *homeomorphism*.

Proof: Scalar multiplication is continuous, so we need only show that the map is *open*. Given $\varepsilon > 0$, by non-discreteness there is x_o in k so that $0 < |x_o| < \varepsilon$. Since V is Hausdorff, there is a neighborhood U of 0 so that $x_o e \notin U$. Shrink U so it is *balanced*. Take $x \in k$ so that $xe \in U$. If $|x| \geq |x_o|$ then $|x_o x^{-1}| \leq 1$, so that

$$x_o e = (x_o x^{-1})(xe) \in U$$

by the balanced-ness of U, contradiction. Thus,

$$xe \in U \implies |x| < |x_o| < \varepsilon \qquad ///$$

Corollary: Fix $x_o \in k$. A not-identically-zero k-linear k-valued function f on V is *continuous* if and only if the affine hyperplane

$$H = \{v \in V : f(v) = x_o\}$$

is *closed* in V.

Proof: For f is continuous, H is closed, being the complement of the open $f^{-1}(\{x \neq x_o\})$. For the converse, take $x_o = 0$, since vector additions are homeomorphisms of V to itself.

For $v_o, v \in V$ with $f(v_o) \neq 0$,

$$f\big(v - f(v)f(v_o)^{-1}v_o\big) \;=\; f(v) - f(v)f(v_o)^{-1}f(v_o) = 0$$

Thus, V/H is one-dimensional. Let $\bar{f} : V/H \to k$ be the induced k-linear map on V/H so that $f = \bar{f} \circ q$:

$$\bar{f}(v + H) = f(v)$$

By the previous proposition, $\bar{f}$ is a homeomorphism to k. so f is continuous. ///

Proof of theorem: To prove the uniqueness of the topology, prove that for any k-basis $e_1, \ldots, e_n$ for V, the map $k \times \ldots \times k \to V$ by

$$(x_1, \ldots, x_n) \to x_1 e_1 + \ldots + x_n e_n$$

is a homeomorphism. Prove this by induction on the dimension n.

$n = 1$ was treated already. Granting this, since k is complete, the lemma asserting the closed-ness of complete subspaces shows that any one-dimensional subspace is closed.

Take $n > 1$, and let $H = ke_1 + \ldots + ke_{n-1}$. By induction, H is closed in V, so V/H is a topological vector space. Let q be the quotient map. V/H is a one-dimensional topological vectorspace over k, with basis $q(e_n)$. By induction,

$$\varphi : xq(e_n) = q(xe_n) \to x$$

is a homeomorphism to k.

Likewise, ke_n is a closed subspace and we have the quotient map

$$q' : V \to V/ke_n$$

We have a basis $q'(e_1), \ldots, q'(e_{n-1})$ for the image, and by induction

$$\phi' : x_1 q'(e_1) + \ldots + x_{n-1} q'(e_{n-1}) \to (x_1, \ldots, x_{n-1})$$

is a homeomorphism.

By induction,

$$v \to (\phi \circ q)(v) \times (\phi' \circ q')(v)$$

is continuous to

$$k^{n-1} \times k \approx k^n$$

On the other hand, by the continuity of scalar multiplication and vector addition, the map

$$k^n \to V \quad \text{by} \quad x_1 \times \ldots \times x_n \to x_1 e_1 + \ldots + x_n e_n$$

is continuous.

The two maps are mutual inverses, proving that we have a homeomorphism.

Thus, a n-dimensional subspace is homeomorphic to k^n, so is complete, since (as follows readily) a finite product of complete spaces is complete.

Thus, by the lemma asserting the closed-ness of complete subspaces, an n-dimensional subspace is always closed.

Continuity of a linear map $f : X \to k^n$ implies that the kernel $N = \ker f$ is closed. On the other hand, if N is closed, then X/N is a topological vectorspace of dimension at most n. Therefore, the induced map $\bar{f} : X/N \to V$ is unavoidably continuous. But then $f = \bar{f} \circ q$ is continuous, where q is the quotient map.

In particular, any k-linear map $V \to V$ has finite-dimensional kernel, so the kernel is closed, and the map is continuous.

This completes the induction. ///

Corollary: Finite field extensions K of complete, non-discrete k have unique Hausdorff topologies making addition and multiplication continuous.

Proof: K is a finite-dimensional k-vectorspace. The only ingredient perhaps not literally supplied by the theorem is the continuity of the multiplication by elements of K. Such multiplications are k-linear endomorphisms of the vector space K, so are continuous, by the theorem. ///

Remark: This discussion still did *not* use *local compactness* of the field k, so is not specifically number theoretic.
