Recap:

Theorem: Two norms $| * |_1$ and $| * |_2$ on k give the same *non-discrete* topology on a field k if and only if $| * |_1 = | * |_2^t$ for some $0 < t \in \mathbb{R}$. [Last time]

Theorem: Over a complete, non-discrete normed field k,
• A *finite-dimensional* k-vectorspace V has just one Hausdorff topology so that vector addition and scalar multiplication are continuous (a *topological vectorspace* topology). All linear endomorphisms are *continuous*.
• A finite-dimensional k-subspace V of a topological k-vectorspace W is necessarily a *closed* subspace of W.
• A k-linear map $\phi : X \to V$ to a finite-dimensional space V is continuous if and only if the kernel is closed.

Corollary: Finite field extensions K of complete, non-discrete k have unique Hausdorff topologies making addition and multiplication continuous.

Constructions/existence: For any Dedekind domain $\mathfrak{o}$, and for a non-zero prime $\mathfrak{p}$ in $\mathfrak{o}$, the $\mathfrak{p}$-adic norm is

$$|x|_\mathfrak{p} \;=\; C^{-\mathrm{ord}_\mathfrak{p} x} \qquad\qquad (\text{where } x\cdot\mathfrak{o} = \mathfrak{p}^{\mathrm{ord}_\mathfrak{p} x}\cdot \text{prime-to-}\mathfrak{p})$$

and $C > 1$ is a constant. Since this norm is ultrametric/non-archimedean, the choice of C does not immediately matter, but it *can* matter in interactions of norms for varying $\mathfrak{p}$, as in the **product formula** for number fields and function fields. Recall the product formula for $\mathbb{Q}$:

$$\prod_{v\le\infty} |x|_v \;=\; 1 \qquad\qquad (\text{for } x\in\mathbb{Q}^\times)$$

That is, with $|*|_\infty$ the 'usual' absolute value on $\mathbb{R}$,

$$|x|_\infty\cdot\prod_{p \text{ prime}} |x|_p \;=\; 1 \qquad\qquad (\text{for } x\in\mathbb{Q}^\times)$$

Recall the *Proof:* Both sides are *multiplicative* in x, so it suffices to consider $x = \pm 1$ and $x = q$ prime. For units ± 1, both sides are 1. For $x = q$ prime, $|q|_\infty = q$, while $|q|_q = 1/q$, and for $p \neq q$, $p < \infty$, $|q|_p = 1$. Thus, both sides are 1. ///

One normalization to have the product formula hold for *number fields* k: for $\mathfrak{p}$ lying over p, letting $k_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of k and Q_p the usual p-adic completion of $\mathbb{Q}$,

$$|x|_\mathfrak{p} \;=\; |N^{k_\mathfrak{p}}_{\mathbb{Q}_p} x|_p$$

For *archimedean* completion k_v of k, define (or renormalize)

$$|x|_v \;=\; |N^{k_v}_{\mathbb{R}} x|_\infty$$

The latter entails a normalization which (harmlessly) fails to satisfy the triangle inequality:

$$|x|_\mathbb{C} \;=\; |N^{\mathbb{C}}_{\mathbb{R}} x|_\infty \;=\; x \cdot \overline{x} \;=\; \textit{square} \text{ of usual complex abs value}$$

This normalization is used only in a multiplicative context, so failure of the triangle inequality is harmless. The metric topology is given by the *usual* norm.

In other words, for primes $\mathfrak{p}$ in $\mathfrak{o}$, in the formula above take $C = N\mathfrak{p} = |\mathfrak{o}/\mathfrak{p}|$, so

$$|x|_{\mathfrak{p}} \;=\; N\mathfrak{p}^{-\mathrm{ord}_{\mathfrak{p}} x}$$

Theorem: *(Product formula for number fields)*

$$\prod_{\text{places } w \text{ of } k} |x|_w \;=\; \prod_{\text{places } v \text{ of } \mathbb{Q}} \; \prod_{w|v} |N^{k_w}_{\mathbb{Q}_v}(x)|_v \;=\; 1 \qquad (\text{for } x \in k^\times)$$

Indeed, reduce to the product formula for $\mathbb{Q}$ by showing

$$\prod_{w|v} N^{k_w}_{\mathbb{Q}_v}(x) \;=\; N^k_{\mathbb{Q}}(x) \qquad (\text{for } x \in k, \text{ abs value } v \text{ of } \mathbb{Q})$$

Proof: Recall that one way to define Galois norm is, for an algebraically closed field Ω containing $\mathbb{Q}$,

$$N^k_{\mathbb{Q}}(x) \;=\; \prod_{\mathbb{Q}-algebra\ maps\ \sigma:k\to\Omega} \sigma(x)$$

Claim: Let Ω be an algebraic closure of $\mathbb{Q}_v$. There is a natural isomorphism of sets

$$\mathrm{Hom}_{\mathbb{Q}-alg}(k,\Omega) \approx \mathrm{Hom}_{\mathbb{Q}_v-alg}(\mathbb{Q}_v \otimes_{\mathbb{Q}} k, \Omega)$$

by

$$\Big(x \to \sigma(x)\Big) \longrightarrow \Big(\alpha \otimes x \to \alpha \cdot \sigma(x)\Big)$$

Proof: Recall that a map from the tensor product is specified by its values on monomials $\alpha \otimes x$, and that these values can indeed be arbitrary, as long as the image of $\alpha a \otimes x$ is the same as that of $\alpha \otimes ax$, for $a \in \mathbb{Q}$.

Then the inverse set-map is

$$\Big(\alpha \otimes x \to \tau(\alpha \otimes x)\Big) \longrightarrow \Big(x \to \tau(1 \otimes x)\Big) \qquad ///$$

Remark: This is an example of *extension of scalars*, an example of a *left adjoint* to a forgetful functor. Then the isomorphism is an example of an *adjunction*.

Next, for finite *separable* $k/\mathbb{Q}$, invoke the theorem of the primitive element to choose α such that $k = \mathbb{Q}(\alpha)$, and let $P \in \mathbb{Q}[x]$ be the minimal polynomial of α over $\mathbb{Q}$. Since $k/\mathbb{Q}$ is separable, P has no repeated roots in an algebraic closure, etc. Then

$$\mathbb{Q}_v \otimes_{\mathbb{Q}} k \;\approx\; \mathbb{Q}_v \otimes_{\mathbb{Q}} \mathbb{Q}[x]/P \;\approx\; \mathbb{Q}_v[x]/P$$

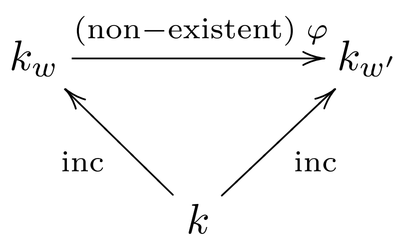
$$\approx\; \coprod_j \mathbb{Q}_v[x]/P_j \;\approx\; \text{coproduct of finite field extensions of } \mathbb{Q}_v$$

by Sun-Ze's theorem, where the P_j are the irreducible factors of P in $\mathbb{Q}_v[x]$, and we use the separability of $k/\mathbb{Q}$ to know that no repeated factors appear. By the defining property of coproducts

$$\mathrm{Hom}_{\mathbb{Q}_v-alg}(\coprod_j \mathbb{Q}_v[x]/P_j,\, \Omega) \;\approx\; \prod_j \mathrm{Hom}_{\mathbb{Q}_v-alg}(\mathbb{Q}_v[x]/P_j,\, \Omega)$$

Because Ω is a field, the $\mathbb{Q}_v$-algebra homs $\mathbb{Q}_v \otimes_{\mathbb{Q}} k \to \Omega$ biject with the maximal ideals of the $\mathbb{Q}_v \otimes_{\mathbb{Q}} k$. The maximal ideals in a product $K_1 \times \ldots \times K_n$ of fields K_j are $M_j = K_1 \times \ldots \times \widehat{K_j} \times \ldots \times K_n$. Thus, the homs to Ω, with kernel M_j, are identified with homs $K_j \to \Omega$. That is, the set of $\mathbb{Q}$-homs $k \to \Omega$ is *partitioned* by the $\mathbb{Q}_v$-homs of the direct summands $\mathbb{Q}_v[x]/P_j$ to Ω.

It remains to show that the direct summands $\mathbb{Q}_v[x]/P_j$ are exactly the completions k_w of k extending the completion $\mathbb{Q}_v$ of $\mathbb{Q}$, *distinct* in the sense that there is *no* topological isomorphism φ fitting into a diagram



(for $w \neq w'$)

First, Ω has a unique topological $\mathbb{Q}_v$-vectorspace topology, because it is an ascending union ((filtered) *colimit*!) of finite-dimensional $\mathbb{Q}_v$-vectorspaces, which have unique topological vector space topologies. Colimits are unique, up to unique isomorphism.

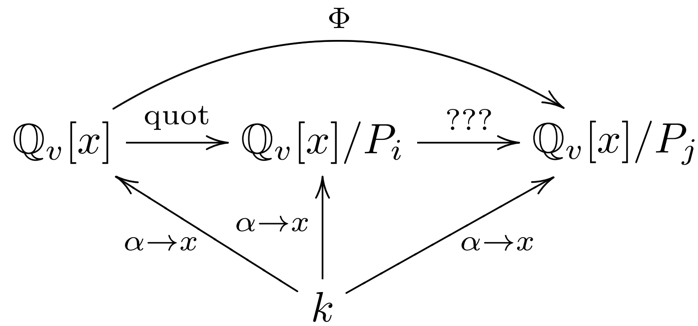
On one hand, $\sigma : k \to \Omega$ (over $\mathbb{Q}$) gives k a Hausdorff topology with continuous addition, multiplication, and non-zero inversion. The compositum $\mathbb{Q}_v \cdot \sigma(k)$ is finite-dimensional over $\mathbb{Q}_v$, so the closure of $\sigma(k)$ in Ω is a *complete* $\mathbb{Q}_v$ topological vector space. Thus, $\sigma : k \to \Omega$ gives a completion of k extending $\mathbb{Q}_v$.

On the other hand, a completion k_w is really an inclusion $k \to k_w$ with k_w complete. Again, there is the adjunction

$$\mathrm{Hom}_{\mathbb{Q}-alg}(k, k_w) \approx \mathrm{Hom}_{\mathbb{Q}_v-alg}(\mathbb{Q}_v \otimes_{\mathbb{Q}} k, k_w)$$

Thus, in fact, $\mathbb{Q}_v[x]/P_j \approx k_w$ for some P_j.

By the separability of $k/\mathbb{Q}$, the P_j's have no common factors, so the inclusions $k \to \mathbb{Q}_v[x]/P_j$ by $\alpha \to x$ mod P_j are incompatible with every non-zero $\mathbb{Q}_v$-hom $\mathbb{Q}_v[x]/P_i \to \mathbb{Q}_v[x]/P_j$ for $i \neq j$. Indeed, the requirement $\alpha \to x$ mod P_j limits the candidates to situations



which forces $\ker\Phi = \langle P_j \rangle$. This cannot factor through the quotient. Thus, there are no isomorphisms among the $\mathbb{Q}_v[x]/P_j$ compatible with the inclusions of k.

In summary, we have proven that the *global* (Galois) norm $N^k_{\mathbb{Q}}$ is the product of the *local* norms, reducing the product formula for number fields to that for $\mathbb{Q}$. ///

Remark: The argument did not depend on the specifics, so applies to extensions K/k and completions k_v of the base field. In the course of the proof, some useful auxiliary points were demonstrated, stated now in general:

Corollary: Let k be a field with completion k_v. Let K be a finite separable extension of k. Let w index the topological isomorphism classes of completions of K extending k_v. The sum of the *local* degrees is the *global* degree:

$$\sum_{w|v}[K_w : k_v] = [K : k]$$

Corollary: For K/k finite separable, the topological isomorphism classes of completions K_w of K extending k_v arise from inclusions of K to the algebraic closure of k_v. (This does not address automorphisms.)

Corollary: The global trace $K \to k$ is the sum of the local traces $K_w \to k_v$.

The following generalizes to number fields and functions fields over finite fields. Traditionally, this result (and its generalizations) are called *Ostrowski's theorem*, but there are some issues surrounding this attribution.

Classification of completions: The topologically (via the associated metrics) inequivalent (non-discrete) norms on $\mathbb{Q}$ are the usual $\mathbb{R}$ norm and the p-adic $\mathbb{Q}_p$'s.

Proof: Let $|*|$ be a norm on $\mathbb{Q}$. It turns out (intelligibly, if we guess the answer) that the watershed is whether $|*|$ is *bounded* or *unbounded* on $\mathbb{Z}$. That is, the statement of the theorem could be sharpened to say: norms on $\mathbb{Q}$ bounded on $\mathbb{Z}$ are topologically equivalent to p-adic norms, and norms unbounded on $\mathbb{Z}$ are topologically equivalent to the norm from $\mathbb{R}$.

To say that $|*|$ is *bounded* on $\mathbb{Z}$, but *not discrete*, implies that $|p| < 1$ for some prime number p, by unique factorization. Suppose that there were a second prime q with $|q| < 1$. Then...

... with $a, b \in \mathbb{Z}$ such that $ap^m + bq^n = 1$ for positive integers m, n,

$$1 \;=\; |1| \;=\; |ap^m + bq^n| \;\le\; |a| \cdot |p|^m + |b| \cdot |q|^n \;\le\; |p|^m + |q|^n$$

This is impossible if *both* $|p| < 1$ and $|q| < 1$, by taking m, n large. Thus, for $|*|$ bounded on $\mathbb{Z}$, there is a unique prime p such that $|p| < 1$. Up to normalization, such a norm is the p-adic norm.

Next, claim that if $|a| \le 1$ for some $1 < a \in \mathbb{Z}$, then $|*|$ is *bounded* on $\mathbb{Z}$. Given $1 < b \in \mathbb{Z}$, write b^n in an a-ary expansion

$$b^n \;=\; c_o + c_1 a + c_2 a^2 + \ldots + c_\ell a^\ell \qquad\qquad (\text{with } 0 \le c_i < a)$$

and apply the triangle inequality,

$$|b|^n \;\le\; (\ell + 1) \cdot \underbrace{(1 + \ldots + 1)}_{a} \;\le\; (n \log_a b + 1) \cdot a$$

Taking n^{th} roots and letting $n \to +\infty$ gives $|b| \le 1$, and $|*|$ is bounded on $\mathbb{Z}$.

The remaining scenario is $|a| \geq 1$ for $a \in \mathbb{Z}$. For $a > 1$, $b > 1$, the a-ary expansion

$$b^n \;=\; c_o + c_1 a + c_2 a^2 + \ldots + c_\ell a^\ell \qquad\qquad (\text{with } 0 \leq c_i < a)$$

with $|a| \geq 1$ gives

$$|b|^n \;\leq\; (\ell + 1) \cdot \underbrace{(1 + \ldots + 1)}_{a} \cdot |a|^\ell \;\leq\; (n \log_a b + 1) \cdot a \cdot |a|^{n \log_a b + 1}$$

Taking n^{th} roots and letting $n \to +\infty$ gives $|b| \leq |a|^{\log_a b}$. Similarly, $|a| \leq |b|^{\log_b a}$. Since $|*|$ is not bounded on $\mathbb{Z}$, there is $C > 1$ such that $|a| = C^{\log |a|}$ for all $0 \neq a \in \mathbb{Z}$. Up to normalization, this is the usual absolute value for $\mathbb{R}$. ///