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Riemann-Hadamard product for $\zeta(s)$

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The zeta function $\zeta(s) = \sum_n 1/n^s$ has an Euler product $\zeta(s) = \prod_{p \text{ prime}} 1/(1-p^{-s})$ on $\mathrm{Re}(s) > 1$. Riemann *anticipated* its factorization in terms of its zeros:

$$(s-1)\zeta(s) \;=\; e^{a+bs} \cdot \prod_\rho \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho} \cdot \prod_{n \ge 1} \Big(1 + \frac{s}{2n}\Big) e^{-s/2n} \qquad (\zeta(\rho) = 0 \text{ with } 0 < \mathrm{Re}(\rho) < 1)$$

[Riemann 1859] equated the two products, and obtained his Explicit Formula relating weighted prime-counting to the complex zeros of zeta.

[Hadamard 1893] confirmed Riemann's surmise about the product in terms of zeros, showing that an entire function f with growth

$$|f(z)| \;\ll_\varepsilon\; e^{|z|^{\lambda+\varepsilon}} \qquad (\text{for all } \varepsilon > 0)$$

possesses a *factorization*[1] for integer h with $h \le \lambda < h+1$:

$$f(z) \;=\; e^{g(z)} \cdot z^m \cdot \prod_j \Big(1 - \frac{z}{z_j}\Big) e^{z/z_j + (z/z_j)^2/2 + (z/z_j)^3/3 + \ldots + (z/z_j)^h/h} \qquad (\text{with } g \text{ polynomial of degree } h)$$

Riemann's factorization follows from Hadamard's theorem with $h = 1$, that is, with *linear* exponents, upon verification that $(s-1)\zeta(s)$ is entire and has growth

$$|(s-1)\zeta(s)| \;\ll_\varepsilon\; e^{|s|^{1+\varepsilon}} \qquad (\text{for all } \varepsilon > 0)$$

The most substantial point in obtaining such an estimate is the Stirling-Laplace asymptotics for $\Gamma(s)$:

$$\Gamma(s) \sim e^{-s} \cdot s^{s-\frac{1}{2}} \cdot \sqrt{2\pi}$$

Riemann's functional equation

$$\pi^{\frac{s}{2}}\, \Gamma\big(\frac{s}{2}\big)\, \zeta(s) \;=\; \pi^{\frac{1-s}{2}}\, \Gamma\big(\frac{1-s}{2}\big)\, \zeta(1-s)$$

and the functional equation $\Gamma(s)\Gamma(1-s) = \pi/\sin \pi s$ are also necessary.

Riemann's integral representation

$$\pi^{-\frac{s}{2}} \Gamma\big(\frac{s}{2}\big) \cdot \zeta(s) \;=\; \int_0^\infty y^{\frac{s}{2}} \sum_{n=1}^\infty e^{-\pi n^2 y}\, \frac{dy}{y} \;=\; \int_1^\infty \big(y^{\frac{s}{2}} + y^{\frac{1-s}{2}}\big) \sum_{n=1}^\infty e^{-\pi n^2 y}\, \frac{dy}{y} + \frac{1}{s-1} - \frac{1}{s}$$

gives an easy useful estimate on the entire function $s(1-s)\pi^{-\frac{s}{2}}\Gamma\big(\frac{s}{2}\big) \cdot \zeta(s)$, as follows.

The sum, is easily estimated:

$$\sum_{n=1}^\infty e^{-\pi n^2 y} \;\le\; \sum_{n=1}^\infty e^{-\pi n y} \;\le\; \frac{e^{-\pi y}}{1 - e^{-\pi y}} \;\ll\; e^{-y} \qquad (\text{for } y \ge 1)$$

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[1] General factorization of entire functions in terms of their zeros is due to Weierstrass [ref?]. Sharper conclusions from growth estimates are due to [Hadamard 1893]. [Riemann 1859]'s presumed existence of a factorization to see the connection between prime numbers and complex zeros of zeta, was a significant impetus to Weierstrass' and Hadamard's study of products in succeeding decades.

By symmetry, take $\mathrm{Re}(s) \ge \frac{1}{2}$, and let $\sigma = \mathrm{Re}(s)$. Then

$$\Big| \int_1^\infty \big(y^{\frac{s}{2}} + y^{\frac{1-s}{2}}\big) \sum_{n=1}^\infty e^{-\pi n^2 y} \, \frac{dy}{y} \Big| \;\le\; \int_0^\infty y^\sigma \, e^{-y} \, \frac{dy}{y} \;=\; \Gamma(\sigma)$$

$$\ll_\delta \; \sigma^{\sigma - \frac{1}{2}} \cdot e^{-\sigma} \;=\; e^{(\sigma - \frac{1}{2}) \log \sigma - \sigma} \;\ll_\varepsilon\; e^{|\sigma - \frac{1}{2}|^{1+\varepsilon}} \;\ll\; e^{|s|^{1+\varepsilon}}$$

Multiplying through by $s(1-s)$ does not disturb this family of estimates, and makes the polar terms $1/(s-1)$, $1/s$ polynomials. Thus,

$$\Big| s(1-s)\pi^{-\frac{s}{2}} \Gamma\big(\frac{s}{2}\big) \cdot \zeta(s) \Big| \;\ll_\varepsilon\; e^{|s|^{1+\varepsilon}}$$

This shows that $\pi^{-s/2}\Gamma(s/2) \cdot \zeta(s)$ admits a product with linear exponential factors, and then the same holds for $\Gamma(s/2) \cdot \zeta(s)$.

We prove that $1/\Gamma(s)$ has an Hadamard product with linear exponentials by proving a similar bound. For $\mathrm{Re}(s) \ge \delta > 0$, from Stirling-Laplace,

$$\Big| \frac{1}{\Gamma(s)} \Big| \;\sim\; \frac{1}{\sqrt{2\pi}} \cdot |e^{-(s - \frac{1}{2}) \log s + s}| \;\ll_\varepsilon\; e^{|s|^{1+\varepsilon}} \qquad (\text{for } \mathrm{Re}(s) \ge \delta > 0)$$

Since $\Gamma(s)$ has no poles in $\mathrm{Re}(s) > 0$, the relation $\pi/\big(\Gamma(s)\Gamma(1-s)\big) = \sin \pi s$ shows $1/\Gamma(s)$ is entire. Also,

$$\Big| \frac{1}{\Gamma(1-s)} \Big| \;=\; \Big| \frac{\Gamma(s) \cdot \sin \pi s}{\pi} \Big| \;\ll_\varepsilon\; e^{|s|^{1+\varepsilon}} \cdot e^{\pi |s|} \;\ll_\varepsilon\; e^{|s|^{1+\varepsilon}} \qquad (\text{for } \mathrm{Re}(s) \ge \delta > 0)$$

Since $|s| \ll_\varepsilon |1-s|^{1+\varepsilon}$ for every $\varepsilon > 0$, we have suitable bounds on $1/\Gamma(s)$ to conclude that it has an Hadamard product with linear exponentials. We know its zeros are non-positive integers, so for some constants a, b

$$\frac{1}{\Gamma(s)} \;=\; e^{a+bs} \cdot s \cdot \prod_{n=1}^\infty \Big(1 + \frac{s}{n}\Big) \, e^{-s/n}$$

Combining this with the product for $\Gamma(s/2) \cdot \zeta(s)$ gives Riemann's presumed product, for some a, b

$$\zeta(s) \;=\; e^{a+bs} \cdot s \cdot \prod_\rho \Big(1 - \frac{s}{\rho}\Big) e^{s/\rho} \cdot \prod_{n \ge 1} \Big(1 + \frac{s}{2n}\Big) e^{-s/2n} \qquad (\zeta(\rho) = 0 \text{ with } 0 < \mathrm{Re}(\rho) < 1)$$

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