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Review examples discussion 00

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-disc-00.pdf]

[00.1] (There is not much hope in making sense of the outcome of an uncountable number of non-zero operations:) Let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω, show that $\sup_F \sum_{\alpha\in F} \alpha = +\infty$.

Discussion: Let $\Omega_1 = \{\omega \in \Omega : \omega > 1\}$, and for $n = 2, 3, \ldots$, let $\Omega_n = \{\omega \in \Omega : \frac{1}{n} < \omega \le \frac{1}{n-1}\}$. There are countably many such sets, so in (at least) one of them Ω_{n_o} there must be infinitely-many elements of Ω (or else Ω would be a countable union of countable sets, hence countable). Then

$$\sup_F \sum_{\alpha\in F} \;\ge\; \sup_{F\subset\Omega_{n_o}} \sum_{\alpha\in F} \;\ge\; \sup_{F\subset\Omega_{n_o}} \#F\cdot\frac{1}{n_o} \;=\; \frac{1}{n_o}\sup_{F\subset\Omega_{n_o}} \#F \;=\; +\infty$$

because Ω_{n_o} is infinite. ///

[00.2] (The *archimedean property* of the real numbers:) Using the characterization of $\mathbb{R}$ as the metric-space completion of $\mathbb{Q}$, show that a real number x with $|x| < \frac{1}{n}$ for $n = 1, 2, 3, \ldots$ must be 0.

Discussion: One aspect of the/a completion is that every Cauchy sequence $\{r_\ell\}$ has a (unique!) limit r_∞, characterized by the property that, for all $1 \le n \in \mathbb{Z}$, there is ℓ_o such that for all $\ell \ge \ell_o$, $|r_\ell - r_\infty| < \frac{1}{n}$.

Let $\{r_\ell\}$ be the constant sequence $r_\ell = x$. This is certainly Cauchy, and 0 satisfies the condition to be the limit: given $1 \le n \in \mathbb{Z}$, take ℓ_o large enough such that $|x_\ell| < \frac{1}{n}$ for every $\ell \ge \ell_o$. Then

$$|x_\ell - 0| \;=\; |x_\ell| \;<\; \frac{1}{n} \qquad\qquad (\text{for } \ell \ge \ell_o)$$

[00.3] Prove carefully that the *inf* of a *finite* set of (strictly) positive real numbers is (strictly) positive.

Discussion: There are several ways to do this, of course, depending on the assumed context. One is as follows. Let the set be $\{r_1, \ldots, r_k\}$. In light of the previous example, it suffices to find $1 \le n \in \mathbb{Z}$ such that $|r_j| \ge \frac{1}{n}$ for $j = 1, \ldots, k$. Since $r_j > 0$, there is $1 \le n_j \in \mathbb{Z}$ such that $\frac{1}{n_j} \le r_j$. The set $\{n_1, \ldots, n_k\}$ is a *finite* set of positive integers, so there is a positive integer n greater all the n_j. Then $\frac{1}{n} \le \frac{1}{n_j} \le r_j$ for all j. ///

[00.4] Prove (or review the proof) that intervals $[a, b] \subset \mathbb{R}$ (with $-\infty < a < b < \infty$) are *connected* in the sense that they cannot be written as a disjoint union of two non-empty (relatively) open subsets. Use this to prove the intermediate value theorem for continuous functions.

Discussion: Suppose $[a, b] = U \cup V$ with disjoint, non-empty open U, V. Since both are non-empty, they have sups and infs. If $\sup U < b$, then u cannot contain $\sup U$, since it would contain a neighborhood of $\sup U$, so $\sup U$ would be larger. Thus, if $\sup U < b$, then $\sup U \in V$. Since V contains a neighborhood of $\sup U$, $U \cap V$ would be non-zero, contradicting the disjointness.

Similarly for $\inf U$. Thus, $\sup U = b$ and $\inf U = a$, and U contains a neighborhood of a and a neighborhood of b. But the same must be true of V, and intersections of neighborhoods of a are neighborhoods of a (and similarly of b), contradicting the disjointness.

To prove the intermediate value theorem from the previous, let f be continuous on $[a, b]$, and suppose f omits value 0. Replacing f by $-f$ if necessary, suppose $f(a) < 0$ and $f(b) > 0$. Consider $U = \sup\{x : f(x) < 0\}$ and $V = \inf\{x : f(x) > 0\}$ exist. As inverse images of opens, these are open. Since f omits the value 0, they are non-empty. But their union would be $[a, b]$, contradicting its connectedness. ///

[00.5] Prove (or review the proof) that a continuous real-valued function f on a finite interval $[a, b] \subset \mathbb{R}$ assumes its *inf*. That is, there is a point $x_o \in [a, b]$ such that $f(x_o) = \inf_{x\in[a,b]} f(x)$.

Discussion: Let $\{x_n : n = 1, 2, \ldots\}$ be such that $f(x_n) \to \inf_{x\in[a,b]} f(x)$. Since $[a, b]$ is compact, it is sequentially compact (consider the opens $U_n = [a, b] - \{x_n\} \subset [a, b]$, ...). Thus, there is a convergent subsequence $\{x_{n_\ell} : \ell = 1, 2, \ldots\}$, and

$$\inf_{x\in[a,b]} f(x) \;=\; \lim_n f(x_n) \;=\; \lim_\ell f(x_{n_\ell}) \;=\; f\Big(\lim_\ell x_{n_\ell}\Big)$$

by continuity of f and convergence of the subsequence. ///

[00.6] Prove (or review the proof) that a continuous real-valued function f on a finite closed interval $[a, b] \subset \mathbb{R}$ is *uniformly* continuous: for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Discussion: Given $\varepsilon > 0$ and $x \in [a, b]$, take $\delta_x > 0$ such that $|x'-x| < 2\delta_x$ implies $|f(x')-f(x)| < \varepsilon/2$. The open intervals $(x-\delta_x, x+\delta_x)$ cover the compact set $[a, b]$, so there is a finite subcover $\{(x_j-\delta_{x_j}, x_j+\delta_{x_j}) : j = 1, \ldots, N\}$. The minimum $\delta = \min_{j=1,\ldots,N} \delta_j$ is positive (see above). For given $x \in [a, b]$, $x \in (x_j-\delta_{x_j}, x_j+\delta_{x_j})$ for some j.

For x' such that $|x' - x| < \delta$, we have $|x' - x_j| \le |x' - x| + |x - x_j| \le \delta + \delta_j \le 2\delta_j$, so $|f(x') - f(x_j)| < \varepsilon/2$, and

$$|f(x') - f(x)| \;\le\; |f(x') - f(x_j)| + |f(x_j) - f(x)| \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon$$

which is the uniform continuity. ///
