(*September 17, 2016*)

Review examples discussion 01

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-disc-01.pdf]

If you want feedback on your write-ups on any of these examples, please get your write-ups by Monday, 19 Sept, 2016.

[01.1] Prove (or review the proof) that a *uniform* pointwise limit of continuous, real-valued functions on $[a, b]$ is continuous.

Discussion: This is the archetype of a three-ε argument. Let the sequence by $\{f_n\}$, and the pointwise limit $f(x) = \lim_n f_n(x)$. Given $\varepsilon > 0$, by the *uniform* pointwise approach to the limit, take n_o large enough so that for all $m, n \geq n_o$, for all $x \in [a, b]$, $|f_m(x) - f_n(x)| < \varepsilon$. Then $|f(x) - f_n(x)| \leq \varepsilon$ for all $x \in [a, b]$, for all $n \geq n_o$. By the uniform continuity of f_{n_o} on $[a, b]$, let $\delta > 0$ so that $|f_{n_o}(x) - f_{n_o}(y)| < \varepsilon$ for all $|x - y| < \delta$. Then

$$|f(x) - f(y)| \;\leq\; |f(x) - f_{n_o}(x)| + |f_{n_o}(x) - f_{n_o}(y)| + |f_{n_o}(y) - f(y)| \;<\; \varepsilon + \varepsilon + \varepsilon$$

as desired. ///

Note: In the latter situation, there is no compulsion to go back and replace ε by $\varepsilon/3$, since it is obviously possible to do so.

[01.2] Prove (or review the proof) of the *Fundamental Theorem of Calculus*: for a *continuous* function f on $[a, b]$, the function $F(x) = \int_a^x f(t)\, dt$ is *continuously differentiable*, and has derivative f. (Use Riemann's integral.)

Discussion: We use the finite additivity property

$$\int_a^c f(x)\, dx \;=\; \int_a^v f(x)\, dx + \int_v^c f(x)\, dx \qquad \text{(for all } v < c \text{ between } a \text{ and } b)$$

Thus,

$$\frac{F(x+\delta) - F(x)}{\delta} - f(x) \;=\; \frac{\int_x^{x+\delta} f(t)\, dt}{\delta} - f(x)$$

By continuity of f, given $\varepsilon > 0$, take $\delta_o > 0$ sufficiently small so that

$$\sup_{y : x \leq y \leq x+\delta_o} |f(y) - f(x)| \;<\; \varepsilon$$

Then

$$\frac{\int_x^{x+\delta} f(t)\, dt}{\delta} - f(x) \;<\; \frac{(f(x) + \varepsilon)\cdot \delta}{\delta} - f(x) \;=\; \varepsilon$$

and, similarly,

$$\frac{\int_x^{x+\delta} f(t)\, dt}{\delta} - f(x) \;>\; \frac{(f(x) - \varepsilon)\cdot \delta}{\delta} - f(x) \;=\; -\varepsilon$$

Thus, given $\varepsilon > 0$, there is $\delta_o > 0$ such that for every $0 < \delta \leq \delta_o$

$$\Big|\frac{F(x+\delta) - F(x)}{\delta} - f(x)\Big| \;<\; \varepsilon$$

(Finding $\delta_o < 0$ for the same inequality is similar.) ///

[01.3] Prove (or review the proof) that for a sequence of real-valued functions f_n on $[0,1]$ approaching f *uniformly* pointwise, $\lim_n \int_0^1 f_n(x)\,dx = \int_0^1 \lim_n f_n(x)\,dx$. (Use Riemann's integral.)

Discussion: Given $\varepsilon > 0$, let n_o be large enough so that for all $n \geq n_o$, for all $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$. Using *linearity* of integrals,

$$\int_a^b f(x)\,dx \;=\; \int_a^b f(x) - f_{n_o}(x)\,dx + \int_a^b f_{n_o}(x)\,dx$$

Upper and lower bounds are obtained from any upper and lower Riemann sums, for any partition $a = x_1 < \ldots < x_n = b$ of the interval:

$$\int_a^b f(x) - f_{n_o}(x)\,dx \;<\; \sum_{j=1}^{n} (x_{j+1} - x_j)\cdot\varepsilon \;=\; (b-a)\cdot\varepsilon$$

and similarly for a lower bound. ///

[01.4] Show that every open subset of $\mathbb{R}$ is a *countable* union of open intervals.

Discussion: Let S be the set. For $s \in S$, since S is open, there is $0 < \delta_s \in \mathbb{Q}$ such that $(s-2\delta_s, s+2\delta_s) \subset S$. By density of $\mathbb{Q}$ in $\mathbb{R}$ there is q_s in the smaller interval $(s-\delta_s, s+\delta_s)$. Certainly $s \in (q_s - \delta_s, q_s + \delta_s)$, and $(q_s - \delta_s, q_s + \delta_s) \subset S$, because for $|t - q_s| < \delta_s$

$$|s-t| \;\leq\; |s - q_s| + |q_s - t| \;<\; \delta + \delta$$

The collection of *all* pairs $(q,\delta) \in \mathbb{Q}\times\mathbb{Q}$ of rationals q, δ is countable, so the subset of (distinct) pairs occuring as q_s, δ_s for $s \in S$ is countable. (Apparently many of the pairs (q,δ) appear as (q_s,δ_s) for many different $s \in S$.) ///

[01.5] Define an *(outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ given by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.

Discussion: Enumerate the rationals as $r_1, r_2, \ldots$. Given $\varepsilon > 0$, let $U_{n,\varepsilon}$ be the interval $(r_n - \frac{\varepsilon}{2^n}, r + \frac{\varepsilon}{2^n})$. The union of these intervals contains $\mathbb{Q}$, and the sum of lengths is $\varepsilon\cdot(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = \varepsilon$.

The Cantor middle-thirds set can be described in terms of base-three expansions, as follows. All real numbers r in $[0,1]$ have (ternary) expansion $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0,1,2\}$. The expansion is unambiguous except for the possibility of coefficients all 2 beyond a certain point, which we exclude by using

$$\frac{2}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \ldots \;=\; 2\cdot\frac{3^{-n}}{1-\frac{1}{3}} \;=\; 2\cdot\frac{3^{1-n}}{3-1} \;=\; 3^{1-n}$$

Then the middle-thirds set C is the set of reals $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0,2\}$ (with the convention excluding endlessly repeating 2's).

Alternatively, the middle-thirds set C is formed as a *nested intersection*, as follows. Let C_1 be $[0,1]$ with the middle third $(\frac{1}{3}, \frac{2}{3})$ removed. Let C_2 be C_1 with the middle third thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ removed, and so on. At each step, the sum of lengths of the remaining intervals is multiplied by $(1 - \frac{1}{3}) = \frac{2}{3}$, and the number of intervals is multiplied by 2. After n middle-third removals, the result C_n is a union of 2^n intervals each of length 3^{-n}. The Cantor middle-thirds set is $C = \bigcap_n C_n$.

Given $\varepsilon > 0$, choose n large enough so that $2^n/3^n < \varepsilon/2$. Cover each of the 2^n intervals of length 3^{-n} making up C_n by an open interval of length $2 \cdot 3^{-n}$. The sum of the lengths of these 2^n open intervals is

$$2^n \cdot (2 \cdot 3^{-n}) \;=\; 2 \cdot (2/3)^n \;<\; 2 \cdot \frac{\varepsilon}{2} \;=\; \varepsilon$$

This exhibits an open cover of C_n with sum of lengths less than ε. Since $C \subset C_n$, this gives such a cover of C itself, as desired. ///

[01.6] (*) Given an enumeration $r_1, r_2, \ldots$ of rational numbers in $[0,1]$, and given a sequence $y_1, y_2, \ldots \to 0$ of real numbers going to 0, construct a sequence of continuous real-valued functions $\{f_n\}$ on $[0,1]$ whose pointwise limit $f(x) = \lim_n f_n(x)$ is y_n on r_n, and is 0 for x irrational.

Discussion: Let t_n be a tent function centered at 0, of height 1, of width $2/n$. We will specify a collection widths $w(1), w(2), \ldots$ so that

$$f_n(x) \;=\; \sum_{i=1}^{n} y_i \cdot t_{w(i)}(x - r_i)$$

will have the desired behavior. First, take $w(n)$ sufficiently small so that the minimum distance between any two of $r_1, \ldots, r_n$ is at least $2 \cdot w(n)$. This prevents any two of the functions $t_{w(i)}(x - r_i)$ from being non-zero simultaneously. Shrink $w(n)$ further, if necessary, so that $w(n) < \frac{1}{n}$.

Let α be a fixed irrational. Given $\varepsilon > 0$, there is n_o such that $n \geq n_o$ implies that $y_n < \varepsilon$, since $y_n \to 0$. Take $n_1 \geq n_o$ sufficiently large so that $|\alpha - r_j| \geq \frac{1}{n_1}$ for $1 \leq j \leq n_o$, so that for $n \geq n_1$, $t_{w(n)}(\alpha - r_j) = 0$ for $1 \leq j \leq n_o$. Thus, for such n,

$$f_n(\alpha) \;=\; \sum_{n_o \leq i \leq n} y_i \cdot t_{w(i)}(\alpha - r_i) \;<\; \varepsilon \sum_{n_o \leq i \leq n} t_{w(i)}(\alpha - r_i) \;\leq\; \varepsilon$$

since the tents are sufficiently narrow so that no two overlap, so that at most a single $t_{w(i)}(\alpha - r_i)$ is non-zero. This holds for every $\varepsilon > 0$. ///

[01.7] (*) Use this to construct a sequence $\{\{f_{mn} : n = 1, 2, \ldots\} : m = 1, 2, \ldots\}$ of sequences such that $\lim_m(\lim_n f_{mn}(x)) = 1$ for rational x, and 0 for irrational x.

Discussion: For a fixed enumeration of the rationals, and for $\varepsilon > 0$, let $y_n(\varepsilon) = 1/n^{\varepsilon}$. Thus, for fixed $\varepsilon > 0$, $y_n(\varepsilon) \to 0$. On the other hand, for each fixed n, $y_n(\varepsilon) \to 1$ as $\varepsilon \to 0$. In the notation of the previous example, let

$$f_{m,n}(x) \;=\; \sum_{i=1}^{n} y_i(\frac{1}{m}) \cdot t_{w(i)}(x - r_i)$$

so $\lim_n f_{m,n}(x)$ is 0 for irrational x, and is $j^{1/m}$ for $x = r_j$. Then $\lim_m(\lim_n f_{m,n}(x)) = 1$ for rational x, and 0 for irrational x. ///

[01.8] (**) Show that for any sequence $\{f_n\}$ of real-valued functions on $[0,1]$ with $0 \leq f_n(x) \leq 1$ for all x, n, and with $f_n(x) \to 1$ for rational x, there are uncountably-many $y \in [0,1]$ with $\limsup f_n(y) = 1$.

Discussion: Fix an enumeration $r_1, r_2, \ldots$ of the rationals, and sequence $\{f_n\}$ of continuous functions such that $\lim_n f_n(r_m) = 1$ for every m. For each j¡ let n_j be such that $|f_n(r_j) - 1| < 2^{-j}$ for every $n \geq n_j$.

Let q_1 be an arbitrary rational, and n_1 sufficiently large so that $|f_n(q_1) - 1| < 2^{-1}$ for all $n \geq n_1$. Let $\delta_1 > 0$ be small enough so that $|f_{n_1}(x) - f_{n_1}(q_1)| < 2^{-1}$ for $|x - q_1| < \delta_1$, and also $\delta_1 < 2^{-1}$.

Choose any two rationals $q'_2 \neq q''_2$ in the interval $(q_1 - \delta_1, q_1 + \delta_1)$. Take n_2 sufficient large so that both $|f_n(q'_2) - 1| < 2^{-2}$ and $|f_n(q''_2) - 1| < 2^{-2}$. Take $\delta_2 > 0$ sufficiently small so that the intervals $(q'_2 - \delta_2, q'_2 + \delta_2)$ and $(q''_2 - \delta_2, q''_2 + \delta_2)$ are disjoint, and lie inside $(q_1 - \delta_1, q_1 + \delta_1)$, and also $\delta_2 < 2^{-2}$.

For both $q_2 = q_2'$ and for $q_2 = q_2''$, choose (corresponding) rationals q_3' and q_3'' in the interval $(q_2 - \delta_2, q_2 + \delta_2)$. Take n_3 sufficient large so that $|f_n(q_3') - 1| < 2^{-3}$ and $|f_n(q_3'') - 1| < 2^{-3}$ for both values of q_2. Take $\delta_3 > 0$ sufficiently small so that the intervals $(q_3' - \delta_3, q_3' + \delta_3)$ and $(q_3'' - \delta_3, q_3'' + \delta_3)$ are disjoint, and lie inside $(q_2 - \delta_2, q_2 + \delta_2)$, and $\delta_3 < 2^{-3}$.

Continuing inductively, at every stage $j = 1, 2, \ldots$, we have 2^j disjoint intervals and n_j sufficiently large so that $|f_n(x) - 1| < 2^{-j}$ for every x in any one of those intervals, grouped in pairs contained in a single one of the 2^{j-1} disjoint intervals at the previous stage.

Thus, we can index the intervals at stage j by sequences $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j$ where each ε_i is in some fixed set $\{1, 2\}$ with two distinct elements, so that the interval $I(\varepsilon_1, \ldots, \varepsilon_j)$ with index $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j$ at stage j has subintervals $I(\varepsilon_1, \ldots, \varepsilon_j, 1)$ and $I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_j, 2)$ at stage $j + 1$.

Thus, for every infinite sequence $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$ with $\varepsilon_j \in \{1, 2\}$, we have a nested sequence of intervals

$$I(\varepsilon_1) \supset I(\varepsilon_1, \varepsilon_2) \supset I(\varepsilon_1, \varepsilon_2, \varepsilon_3) \supset \ldots$$

with the length of $I(\varepsilon_1, \ldots, \varepsilon_j)$ being 2^{-j}. In particular, for each of the uncountably many $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots)$, the sequence of midpoints is Cauchy, so has a limit x_ε.

By construction, $|f_{n_j}(x_\varepsilon) - 1| < 2^{-j}$ for every j, for every sequence of choices ε. Thus, $\limsup_n f_n(x_\varepsilon) \geq 1$ for every one of the uncountably-many sequences of choices ε. ///

Remarks: Thus, apparently, the characteristic function (also called *indicator function*) of the rationals is *cannot be* the pointwise limit of a sequence of continuous functions. However, the previous examples show that it *is* the pointwise limit of a sequence of pointwise limits of sequences of continuous functions. Apparently, it is not possible to dodge this by any sort of diagonal trick played on sequences of sequences, etc.

This sort of hierarchy of types of functions obtained as pointwise limits was a significant object of research in the late 19th and early 20th century, with hopes for some sort of definitive classification scheme. Eventually, two things became apparent: first, that any classification scheme was too complicated to be useful in ordinary situations, and, second, that many issues turned out to not really depend on any such classification.

For example, rather than insisting on pointwise limits, Lebesgue's notion of pointwise-off-a-set-of-measure-zero did successfully collapse such sequences-of-sequences, simplifying our picture of *functions*. From that viewpoint, the characteristic function of $\mathbb{Q}$ is already almost-everywhere 0, so is operationally 0, and there's nothing needing to be done.
