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Examples 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-disc-02.pdf]

[02.1] Prove that every open in $\mathbb{R}^2$ is a countable union of Cartesian products $(a,b)\times(c,d)$ of open intervals.

Discussion: For a point u in (non-empty) open $U \subset \mathbb{R}^2$, there is an open ball about u of some positive radius and contained in U. Shrink the ball slightly to have positive *rational* radius r. Within distance $r/4$ of u there is a point $(x,y) \in \mathbb{Q}^2$. Then the open ball B_u of radius $r/4$ at (x,y) contains u. Thus, the square $(x-\frac{r}{2}, x+\frac{r}{2}) \times (y-\frac{r}{2}, y+\frac{r}{2})$ contains u. There are only countably-many possibilities for the intervals $(x-\frac{r}{2}, x+\frac{r}{2})$ and $(y-\frac{r}{2})$, so the set of such occuring for $u \in U$ is countable. ///

[02.2] Define tent functions of (half-) width w and height h centered at 0 by

$$t_{w,h}(x) \;=\; \begin{cases} 0 & (\text{for } x \le -w) \\ \frac{h}{w}\cdot(x+w) & (\text{for } -w \le x \le 0) \\ h - \frac{h}{w}\cdot x & (\text{for } 0 \le x \le w) \\ 0 & (\text{for } x \ge w) \end{cases}$$

Show that the functions $f_n(x) = t_{\frac{1}{n},n}(x-\frac{1}{n})$, a sequence of narrowing tents just to the right of 0, go to 0 *pointwise* (everywhere!), but that

$$\lim_n \int_{\mathbb{R}} f_n(x)\cdot g(x)\,dx \;=\; g(0) \qquad (\text{for all } g \in C^o(\mathbb{R}))$$

Discussion: Certainly $f_n(x) = 0$ for $x \le 0$, for all n. For $x > 0$, there is positive integer n_o such that $x > \frac{2}{n_o}$, by the Archimedean property of the reals. Then for $n \ge n_o$ the functions f_n are 0 at x, giving the pointwise convergence to 0.

By design, the integrals of all the f_n are 1. Given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|g(x) - g(0)| < \varepsilon$ for $|x - 0| < \delta$. For n large enough so that $\frac{2}{n} < \delta$,

$$\Big|\int_{\mathbb{R}} f_n(x)\cdot g(x)\,dx \;-\; g(0)\Big| \;=\; \Big|\int_{\mathbb{R}} f_n(x)\cdot\big(g(x)-g(0)\big)\,dx\Big| \;=\; \int_0^{2/n} f_n(x)\cdot\big|g(x)-g(0)\big|\,dx$$

$$\le \int_0^{2/n} f_n(x)\cdot\varepsilon\,dx \;=\; \varepsilon\cdot\int_0^{2/n} f_n(x)\,dx \;=\; \varepsilon$$

giving the convergence of the integrals to $g(0)$ as claimed. ///

[02.3] Show that the functions $f_n(x) = t_{\frac{1}{n},n^2}(x-\frac{1}{n}) - t_{\frac{1}{n},n^2}(x+\frac{1}{n})$, whose graphs are tall tents of area n upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

$$\lim_n \int_{\mathbb{R}} f_n(x)\cdot g(x)\,dx \;=\; 2\cdot g'(0) \qquad (\text{for differentiable } g \text{ with derivative } g' \text{ in } C^o(\mathbb{R}))$$

(Thanks to J. Morey for finding an error in the original computation, which lost the factor of 2...)

Discussion: Certainly $f_n(x) = 0$ for $|x| \geq \frac{2}{n}$, for all n, and $f_n(0) = 0$ for all n. Given $x \neq 0$, there is positive integer n_o such that $|x| > \frac{2}{n_o}$, by the Archimedean property of the reals. Then for $n \geq n_o$ the functions f_n are 0 at x, giving the pointwise convergence to 0.

By a Taylor-Maclaurin expansion, $g(x) = g(0) + g'(0) \cdot x + h(x)$ where $|h(x)/x| \to 0$ as $x \to 0$. By linearity of integrals,

$$\int_{\mathbb{R}} f_n(x) \cdot g(x)\, dx \;=\; \int_{\mathbb{R}} f_n(x) \cdot g(0)\, dx \;+\; \int_{\mathbb{R}} f_n(x) \cdot g'(0)x\, dx \;+\; \int_{\mathbb{R}} f_n(x) \cdot h(x)\, dx$$

$$=\; g(0) \int_{\mathbb{R}} f_n(x)\, dx \;+\; g'(0) \int_{\mathbb{R}} f_n(x) \cdot x\, dx \;+\; \int_{\mathbb{R}} f_n(x) \cdot h(x)\, dx$$

$$=\; g(0) \cdot 0 \;+\; g'(0) \int_{\mathbb{R}} f_n(x) \cdot x\, dx \;+\; \int_{\mathbb{R}} f_n(x) \cdot h(x)\, dx \;=\; g'(0) \int_{\mathbb{R}} f_n(x) \cdot x\, dx \;+\; \int_{\mathbb{R}} f_n(x) \cdot h(x)\, dx$$

Much as in the previous example, for the integral involving h, given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|h(x)/x| < \varepsilon$ for $0 < |x - 0| < \delta$. For n large enough so that $\frac{2}{n} < \delta$,

$$\Big| \int_{\mathbb{R}} f_n(x) \cdot h(x)\, dx \Big| \;=\; \int_{\mathbb{R}} |f_n(x)| \cdot |h(x)|\, dx \;=\; \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot |h(x)|\, dx \;\leq\; \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot \varepsilon |x|\, dx$$

$$=\; \varepsilon \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot |x|\, dx$$

Thus, evaluation of $\int f_n \cdot x$ will also facilitate showing $\int f_n \cdot h \to 0$. Since both f_n and x are *odd*, their product is *even*, so

$$\int_{|x| \leq \frac{2}{n}} f_n(x) \cdot x\, dx \;=\; 2 \cdot \int_0^{2/n} f_n(x) \cdot x\, dx \;=\; 2 \cdot \int_0^{1/n} n^3 x \cdot x\, dx \;+\; 2 \cdot \int_{1/n}^{2/n} (n^2 - n^3(x - \frac{1}{n})) \cdot x\, dx$$

Replacing x by $x + \frac{1}{n}$ in the second integral gives some simplification:

$$2 \cdot \int_0^{1/n} n^3 x \cdot x\, dx \;+\; 2 \cdot \int_0^{1/n} (n^2 - n^3 x) \cdot (x + \frac{1}{n})\, dx \;=\; 2 \cdot \int_0^{1/n} n^2 x + (n - n^2 x)\, dx \;=\; 2 \int_0^{1/n} n\, dx \;=\; 2$$

Thus, $|\int f_n \cdot h| < \varepsilon \cdot 1$ for $n \geq 2/\delta$, so that integral goes to 0, and the coefficient of $g'(0)$ is 2. ///

[02.4] Show that the closed unit ball in ℓ^2, although *closed* and *bounded*, is *not compact*, by showing it is not *sequentially* compact.

Discussion: Let $e_n = (0, \ldots, 0, 1, 0, \ldots)$ with the single 1 at the n^{th} place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of e_n's has no Cauchy subsequence, so no convergent subsequence. ///

[02.5] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals ε_n,

$$C(\varepsilon) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \leq \varepsilon_n\}$$

is compact if and only if $\sum_n |\varepsilon_n|^2 < \infty$.

Discussion: Probably better to rewrite $C(\varepsilon)$ as $C(\delta)$ with $\delta = (\delta_1, \delta_2, \ldots)$. Use the *total boundedness* criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is n_o large enough so that $\sum_{n \geq n_o} \delta_n^2 < \varepsilon^2$. The set

$$C_{n_o} \;=\; \{(z_1, z_2, \ldots, z_{n_o}) \in \mathbb{R}^{n_o} : |z_n| \leq \delta_n\}$$

is a compact subset of $\mathbb{R}^{n_o}$, so certainly has a finite cover by open balls of radius ε. Let the centers of these balls be $w_1, \ldots, w_N$. Let $j : \mathbb{R}^{n_o} \to \ell^2$ be the inclusion $j(z_1, \ldots, z_{n_o}) = (z_1, \ldots, z_{n_o}, 0, 0, \ldots)$. Then we claim that the open balls of radius 2ε at $j(w_1), j(w_2), \ldots, j(w_N)$ cover $C(\delta)$. Indeed, given $z = (z_1, z_2, \ldots) \in C(\delta)$, write $z = j(z') + z''$ where $z' = (z_1, \ldots, z_{n_o})$ and $z'' = z - j(z') = (0, \ldots, 0, z_{n_o+1}, \ldots)$. There is at least one of the w_js within ε of z': let w_{j_o} be such. By the triangle inequality for the norm $|\cdot|_{\ell^2}$ on ℓ^2,

$$d(z, j(w_{j_o})) \;=\; |z - j(w_{j_o})|_{\ell^2} \;=\; |j(z') + z'' - j(w_{j_o})|_{\ell^2} \;\leq\; |j(z') - j(w_{j_o})|_{\ell^2} + |z''|_{\ell^2}$$

$$= \; |z' - w_{j_o}|_{\mathbb{R}^{n_o}} + |z''|_{\ell^2} \;<\; \varepsilon + \varepsilon$$

Thus, $C(\delta)$ can be covered by finitely-many open balls of radius 2ε. ///

[02.6] Show that a closed interval $[a, b]$ has the expected Lebesgue measure, namely, $|b - a|$, by showing that the *inf* of $\sum_{j=1}^n |b_n - a_n|$ for all finite open covers $[a, b] \subset \bigcup_{j=1}^n (a_j, b_j)$ is $|b - a|$.

Discussion: There is no need to use more than one copy of a given interval in approximating the inf, so we can assume the intervals are distinct. Further, we can assume that the finite cover is *minimal* in the sense that no (a_i, b_i) is redundant.

Renumbering if necessary, suppose that $a_1 \leq a_i$ for all $i = 1, \ldots, n$. Necessarily $a_1 < a$, or else the left endpoint of $[a, b]$ is not covered. Without loss of generality, $b_1 > a$, or else we could have dropped (a_1, b_1) from the cover. Then there is at least one index $j > 1$ such that $a_j < b_1$, or else (a_1, b_1) and $\bigcup_{j>1}(a_j, b_j)$ are disjoint, so $[a, b] \cap (a_1, b_1)$ and $[a, b] \cap \bigcup_{j>1}(a_j, b_j)$ are disjoint and non-empty, contradicting the connectedness of $[a, b]$. Renumbering if necessary, $j = 2$. Then $b_2 > b_1$, or else we could have dropped (a_2, b_2) from the cover. In fact, $a_2 > a_1$, or else (a_1, b_1) is redundant.

Suppose by induction, we have renumbered so that $a_1 < a_2 < \ldots < a_m$, $b_1 < \ldots < b_m$, $a_{i+1} < b_i$ for $i = 1, \ldots, m-1$, and $a_1 < a$. If $b_m > b$, we are already done. For $b_m \leq b$, we claim there is an index $j > m$ such that $a_j < b_m$. If not, $\bigcup i \leq m(a_i, b_i)$ and $\bigcup j > m(a_j, b_j)$ are disjoint, meeting $[a, b]$ in non-empty open subsets, contradicting the connectedness of $[a, b]$. Renumber if necessary so that $j = m + 1$. Necessarily $b_{m+1} > b_m$, or else (a_{m+1}, b_{m+1}) was redundant. Thus, by induction, we have $a_1 < a_2 < \ldots < a_n$, $b_1 < \ldots < b_n$, $a_{i+1} < b_i$ for $i = 1, \ldots, m-1$, $a_1 < a$, and $b_n > b$. Then

$$\sum_{i=1}^n b_i - a_i \;=\; \sum_{i=1}^{n-1} (b_i - a_i) \;+\; (b_n - a_n) \;\geq\; \sum_{i=1}^{n-1} (a_{i+1} - a_i) \;+\; (b_n - a_n)$$

$$= \; (a_2 - a_1) + (a_3 - a_2) + \ldots + (a_n - a_{n-1}) + (b_n - a_n) \;=\; b_n - a_1 \;\geq\; b - a$$

as claimed. ///

[02.7] Let f be a continuous function on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. Show that $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive Lebesgue measure.

Discussion: By the intermediate value theorem, that set is not empty. It is the inverse image of an open set, so by continuity is open. Thus, it contains a non-empty interval, which has positive measure. ///

[02.8] Analogous to the Cantor middle-thirds set, but with shrinking ratios of what's removed, form a subset of $[0, 1]$ by first removing the middle $1/4$ of $[0, 1]$, then the middle $1/9$s of the remaining intervals, then the middle $1/16$'s of the remaining intervals, then the middle $1/25$'s of the remaining intervals, ... Show that the nested intersection of these sets has *positive* measure.

(Correction to the original needless specific claim: the measure is $1/2\pi$*.)*

Discussion: At the n^{th} stage, the total of the lengths of the remaining intervals is multiplied by $(1 - \frac{1}{n^2})$. Thus, by the n^{th} stage, the length of the remaining intervals is

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \ldots (1 - \frac{1}{n^2})$$

For $0 < x < \frac{1}{2}$, there is a constant C such that $|\log(1-x)| \le Cx$. Then the logarithm of the finite product is estimated by

$$\Big| \log \Big((1 - \frac{1}{2^2}) \ldots (1 - \frac{1}{n^2} Big| \;\le\; |\log(1 - \frac{1}{2^2})| + \ldots + |\log(1 - \frac{1}{n^2})| \;\le\; \sum_{m=2}^{n} \frac{1}{m^2} \;<\; +\infty$$

Thus, the limit of the finite products is a (finite) non-zero number. ///

[The following is needlessly complicated, as the infinite product actually telescopes. Corrected the constants...]
If we recall that

$$\prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big) \;=\; \frac{\sin \pi z}{\pi z}$$

then

$$\prod_{n=2}^{\infty} \Big(1 - \frac{z^2}{n^2}\Big) \;=\; \frac{\sin \pi z}{(1 - z^2) \cdot \pi z}$$

The left-hand side evaluated at $z = 1$ is the desired value. Near $z = 1$ there is the power series expansion

$$\sin \pi z \;=\; (\sin \pi) - (\cos \pi) \cdot (z - 1) + \ldots \;=\; -(z - 1) + \ldots$$

so *(with missing* π *inserted, cancelling the* π *in the denominator)*

$$\prod_{n=2}^{\infty} \Big(1 - \frac{1}{n^2}\Big) \;=\; \frac{\sin \pi z}{(1 - z^2) \cdot \pi z}\Big|_{z=1} \;=\; \frac{-(z-1)\pi + \ldots}{(1 - z) \cdot (1 + z) \cdot \pi z}\Big|_{z=1} \;=\; \frac{\pi + \ldots}{(1 + 1) \cdot \pi} \;=\; \frac{1}{2}$$
