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Examples discussion 03

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-disc-03.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, 12 Oct 2016.

[03.1] Show that ℓ^2 is *complete* as a metric space.

Discussion: We can do this directly, although it is also a special case of the general fact that $L^2(X,\mu)$ is complete. Indeed, the argument will be a somewhat simpler version of the more general proof.

Let $f_1, f_2, \ldots$ be a Cauchy sequence in ℓ^2. Let $f(n)$ be the n^{th} component of $f \in \ell^2$, for $n = 1, 2, \ldots$. For any $f \in \ell^2$, certainly $|f(n)| \le |f|_{\ell^2}$, so for each n the scalar sequence $f_1(n), f_2(n), f_3(n), \ldots$ must be Cauchy, thus has a limit $f(n)$. We claim that $f = (f(1), f(2), f(3), \ldots)$ is in ℓ^2, and is the ℓ^2 limit of the f_i.

Given $\varepsilon > 0$, there is N sufficiently large so that $|f_i - f_j|_{\ell^2} < \varepsilon$ for all $i, j \ge N$. By a discrete version of Fatou's lemma, for $i \ge N$,

$$\sum_n |f(n) - f_i(n)|^2 \;=\; \sum_n \lim_j |f_j(n) - f_i(n)|^2 \;=\; \sum_n \liminf_j |f_j(n) - f_i(n)|^2 \;\le\; \liminf_j \sum_n |f_j(n) - f_i(n)|^2$$

$$\le\; \liminf_j |f_j - f_i|^2_{\ell^2} \;\le\; \liminf_j \varepsilon^2 \;=\; \varepsilon^2$$

Thus, $f - f_i \in \ell^2$, so $f = (f - f_i) + f_i \in \ell^2$. Then the previous computation shows that for given ε for $i \ge N$ we have $|f - f_i| \le \varepsilon$. Thus, $f_i \to f$ in ℓ^2. ///

Discrete version of Fatou's Lemma: We claim that for $[0, +\infty]$-valued functions f_j on $\{1, 2, 3, \ldots\}$,

$$\sum_{n=1}^\infty \liminf_j f_j(n) \;\le\; \liminf_j \sum_{n=1}^\infty f_j(n)$$

Proof: Letting $g_j(n) = \inf_{i \ge j} f_j(n)$, certainly $g_j(n) \le f_j(n)$ for all n, and $\sum_n g_j(n) \le \sum_n f_j(n)$. Also, $g_1(n) \le g_2(n) \le \ldots$ for all n, and $\lim_j g_j(n) = \liminf_j f_j(n)$. A discrete form of the Monotone Convergence Theorem, proven just below, is

$$\sum_n \lim_j g_j(n) \;=\; \lim_j \sum_n g_j(n)$$

Thus,

$$\sum_n \liminf_j f_j(n) \;=\; \sum_n \lim_j g_j(n) \;=\; \lim_j \sum_n g_j(n) \;=\; \liminf_j \sum_n g_j(n) \;\le\; \liminf_j \sum_n f_j(n)$$

as claimed. ///

Similarly, we have

Discrete version of Lebesgue's Monotone Convergence Theorem: For $[0, +\infty]$-valued functions f_j on $\{1, 2, 3, \ldots\}$, with $f_1(n) \le f_2(n) \le \ldots$ for all n,

$$\lim_j \sum_{n=1}^\infty f_j(n) \;=\; \sum_{n=1}^\infty \lim_j f_j(n) \qquad \text{(allowing value } +\infty)$$

Proof: Each non-decreasing sequence $f_1(n) \leq f_2(n) \leq \ldots$ has a limit $f(n) \in [0, +\infty]$. Similarly, since $\sum_n f_j(n) \leq \sum_n f_{j+1}(n)$, the non-decreasing sequence of these sums has a limit $S = \lim_j \sum_n f_j(n)$. Since $f_j(n) \leq f(n)$, certainly $\sum_n f_j(n) \leq \sum_n f(n)$, and $S \leq \sum_n f(n)$.

Fix N, and put $g(n) = f(n)$ for $n \leq N$ and $g(n) = 0$ for $n > N$. For $\varepsilon > 0$, let

$$E_j \;=\; \{n : \sum_n f_j(n) \geq (1-\varepsilon) \cdot \sum_n g(n)\} \qquad \text{(for } j = 1, 2, \ldots)$$

Certainly $E_1 \subset E_2 \subset \ldots$, since $f_{j+1}(n) \geq f_j(n)$ for all n. We claim that $\bigcup E_j = \{1, 2, \ldots\}$: for $f(n) > 0$,

$$\lim_j f_j(n) \;=\; f(n) \;>\; (1-\varepsilon) \cdot f(n) \;\geq\; (1-\varepsilon) \cdot g(n) \qquad \text{(for all } n)$$

and for $f(n) = 0$, also $g(n) = 0$, and

$$f_1(n) \;\geq\; 0 \;\geq\; (1-\varepsilon) \cdot g(n)$$

Then

$$\sum_n f_j(n) \;\geq\; \sum_{n \in E_j} f_j(n) \;\geq\; (1-\varepsilon) \cdot \sum_{n \in E_j} g(n)$$

The set of n for which $g(n)$ is non-zero is finite, so there is j_o such that for $j \geq j_o$

$$\sum_{n \in E_j} g(n) \;=\; \sum_n g(n) \qquad \text{(for all } j \geq j_o)$$

That is, $\lim_j \sum_n f_j(n) \geq (1-\varepsilon) \sum_n g(n)$. Then

$$S \;=\; \lim_j \sum_n f_j(n) \;\geq\; (1-\varepsilon) \cdot \lim_j \sum_{n \in E_j} g(n) \;=\; (1-\varepsilon) \cdot \sum_n g(n)$$

This holds for every $\varepsilon > 0$, so $S \geq \sum_n g(n) = \sum_{n \leq N} f(n)$. This holds for every N, so $S \geq \sum_n f(n)$. ///

[03.2] Show that the characteristic function χ_E of a measurable set E is measurable.

Discussion: For non-empty open $U \subset \mathbb{R}$, $\chi_E^{-1}(U)$ is the measurable set ϕ if U does not contain either 0 or 1. If $U \ni 1$ but $U \not\ni 0$, then $\chi_E^{-1}(U) = E$, which is measurable. If $U \ni 0$ but $U \not\ni 1$, then $\chi_E^{-1}(U) = E^c$, the complement of E, which is measurable. If U contains both 0 and 1, then $\chi_E^{-1}(U)$ is the whole domain space, which is measurable. ///

[03.3] Show that the product of two $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ is measurable.

Discussion: Let f, g be measurable functions. Let $\Delta : \mathbb{R} \to \mathbb{R}^2$ by $\Delta(x) = (x, x)$, $s : \mathbb{R}^2 \to \mathbb{R}$ by $m(x, y) = x \cdot y$, and $f \oplus g : \mathbb{R}^2 \to \mathbb{R}^2$ by $(f \oplus g)(x, y) = (f(x), g(y))$. Clearly $m \circ (f \oplus g) \circ \Delta = f \cdot g$, and $(f \cdot g)^{-1} = \Delta^{-1} \circ (f \oplus g)^{-1} \circ m^{-1}$.

For open $U \subset \mathbb{R}$, $m^{-1}(U) \subset \mathbb{R}^2$ is open, because m is continuous. Since $\mathbb{R}^2$ is countably based, and in fact has a countable basis consisting of rectangles with rational endpoints, so $m^{-1}(U)$ is a countable unions of rectangles $(a_i, b_i) \times (c_i, d_i)$. Then

$$(f \oplus g)^{-1} \circ m^{-1}(U) \;=\; (f \oplus g)^{-1}(\bigcup_i (a_i, b_i) \times (c_i, d_i))$$

$$= \bigcup_i (f \oplus g)^{-1}((a_i, b_i) \times (c_i, d_i)) \;=\; \bigcup_i f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)$$

The sets $f^{-1}(a_i,b_i) \subset \mathbb{R}$ and $g^{-1}(c_i,d_i) \subset \mathbb{R}$ are Borel sets, so their product is a Borel set in $\mathbb{R}^2$. Then

$$\Delta^{-1}(E_1 \times E_2) \;=\; E_1 \cap E_2 \qquad\qquad (\text{for } E_1, E_2 \text{ measurable in } \mathbb{R})$$

is measurable. ///

[03.4] Show that the set $[0,+\infty] = [0,+\infty) \cup \{\infty\}$ with basis consisting of intervals $[0,b)$, (a,b) for all real $0 < a < b$, and of sets $(b,+\infty) \cup \{\infty\}$ gives a *topology* on $[0,+\infty]$. Show that with this topology $[0,+\infty]$ is Hausdorff, compact, and countably-based.

Discussion: When specifying a topology by giving a *basis*, the only things needed to check are that finite intersections of the basis elements are expressible as unions of the basis sets, and that the whole space is a union of the basis sets.

Indeed, finite intersections of the given sets are again of the same form, that is, are again members of the basis. The whole space is among the basis elements. Thus, this really does specify a topology.

The Haudorff-ness follows from the Hausdorff-ness of $\mathbb{R}$, together with explicit disjoint neighborhoods of $x \neq \infty$ and ∞, namely, $(x,x+1)$ and $(x+2,+\infty) \cup \{\infty\}$ for $x > 0$, and $[0,1)$ and $(2,+\infty) \cup \{\infty\}$ for $x = 0$. The countable basis arises from the countable basis for $\mathbb{R}$, and countable local basis $(a,+\infty) \cup \{\infty\}$ with $0 < a \in \mathbb{Q}$ at ∞.

For compactness, let $C = \{U_\alpha : \alpha \in A\}$ be an open cover. Some U_{α_o} contains ∞, so contains some basis element $(a,\infty) \cup \{\infty\}$. The finite interval $[0,a+1]$ is compact, and C is an open cover of it, so there is a finite subcover $U_{\alpha_1}, \ldots, U_{\alpha_n}$. Then $U_{\alpha_o}, U_{\alpha_1}, \ldots, U_{\alpha_n}$ is a finite subcover of $[0,\infty]$. ///

[03.5] Let f be a $[0,+\infty]$-valued measurable function on $\mathbb{R}$. Show that there is a sequence of non-negative real-valued *simple* functions f_n approaching f pointwise almost-everywhere, so that $\int f_n \to \int f$.

Discussion:

[03.6] Use Urysohn's lemma to prove that $C^o[a,b]$ is dense in $L^1[a,b]$.

Discussion: By the Lebesgue definition of integrals, *simple* functions are dense in $L^1[a,b]$, so it suffices to show that *simple* functions can be well approximated by continuous functions. Granting ourselves the *(outer and inner) regularity* of Lebesgue measure μ, for measurable E there are open U and compact K such that $K \subset E \subset U$, and $\mathfrak{m}(U) - \mu(K) < \varepsilon$. Invoke Urysohn to make a continuous function f taking values in $[0,1]$ and $f|_K = 1$ and $f = 0$ off U. Then

$$\int_a^b |f - \mathrm{ch}_E| \;=\; \int_K |f - \mathrm{ch}_E| + \int_{E-K} |f - \mathrm{ch}_E| + \int_{U-E} |f - \mathrm{ch}_E| \;\leq\; \int_K |1-1| + \int_{E-K} 1 + \int_{U-E} 1$$

$$=\; \mu(E-K) + \mu(U-E) \;=\; \mu(U-K) \;<\; \varepsilon$$

as desired. ///

(I owe you a fuller discussion of regularity of some form of Lebesgue measure!)