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Examples discussion 04

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[04.1] Comparing L^p spaces. Let $1 \le p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

Discussion: Take $p < p'$. We claim that $L^p[a,b] \supset L^{p'}[a,b]$, with proper containment. The function f that is $(x-a)^{-\frac{1}{p'}}$ on $(a,b]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p. Given $f \in L^{p'}[a,b]$, let E be the set of $x \in [a,b]$ where $|f(x)| \ge 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On E, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in L^p[a,b]$. ///

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \ne p'$. Take $1 \le p < p'$. On one hand, $1/(1+|x|)^{1/p'+\varepsilon}$ is in $L^{p'}$ for all $\varepsilon > 0$, but not in L^p for ε small enough so that $\frac{1}{p'} + \varepsilon < \frac{1}{p}$. On the other hand, the function f that is $x^{-\frac{1}{p'}}$ on $(0,1]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p.

We claim that for $1 \le p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f(n) = 1/n^p$ is not in $\ell^{p'}$, but is in ℓ^p. Let $E = \{n \in \{1,2,\ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the *complement* of E is finite, and if $\sum_{n\in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^{p'}$ for $n \in E$, and the complement of E is finite, so $\sum_{n\in E} |f(n)|^{p'} < \sum_{n\in E} |f(n)|^p$, and $f \in \ell^{p'}$. ///

[04.2] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

Discussion: This is a corollary of Jensen's inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1,2,\ldots,n\}$ with measure $\mu(i) = w_i$, and function $f(i) = \log a_i$. Then Jensen's inequality is

$$\exp\Big(\sum_{i=1}^n w_i \cdot \log a_i\Big) \;=\; \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. ///

[04.3] In ℓ^2, show that the point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

Discussion: The minimum principle assures that there is a *unique* closest point w in the closed unit ball B to v, because B is convex, closed, non-empty, and v is not in B.

Suppose w is closer than $v/v|$. Then

$$|v|^2 - 2|v| + 1 \;=\; |v - \frac{v}{|v|}|^2 \;>\; |v-w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + |w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + 1$$

Thus,

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle$$

Thus, the sum of the two inner products is *positive*, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle \;=\; |\langle v,w\rangle + \langle w,v\rangle| \;\le\; 2|v|\cdot|w|$$

Thus, $1 < |w|$, which is impossible. ///

[04.4] For a measurable set $E \subset [0, 2\pi]$, show that

$$\lim_{n\to\infty} \int_E \cos nx \, dx \;=\; 0 \;=\; \lim_{n\to\infty} \int_E \sin nx \, dx$$

Discussion: This is an instance of a *Riemann-Lebesgue lemma*, namely, that Fourier coefficients of an L^2 function on $[0, 2\pi]$ go to 0. Here, the L^2 function is the characteristic function of E, and we use sines and cosines instead of exponentials. ///

[04.5] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. Write out the conclusion of Parseval's theorem for this function.

Discussion: We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0, 2\pi]$. The Fourier coefficients are determined by Fourier's formula

$$\widehat{f}(n) \;=\; \int_0^{2\pi} f(x) \, \frac{e^{-inx}}{\sqrt{2\pi}} \, dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\widehat{f}(n) \;=\; \Big[f(x) \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \Big]_0^{2\pi} - \int_0^{2\pi} 1 \cdot \frac{e^{-inx}}{\sqrt{2\pi} \cdot (-in)} \, dx$$

$$= \Big((\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)}) - (-\pi \cdot \frac{1}{\sqrt{2\pi} \cdot (-in)}) \Big) - 0 \;=\; \frac{2\pi}{\sqrt{2\pi} \cdot (-in)} \;=\; \frac{\sqrt{2\pi}}{-in}$$

The L^2 norm of f is

$$\int_0^{2\pi} (x - \pi)^2 \, dx \;=\; \Big[\frac{(x-\pi)^3}{3} \Big]_0^{2\pi} \;=\; \frac{\pi^3 - (-\pi)^3}{3} \;=\; \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n \neq 0} \Big| \frac{\sqrt{2\pi}}{-in} \Big|^2 \;=\; \frac{2\pi^3}{3}$$

This simplifies first to

$$2 \sum_{n \geq 1} \frac{2\pi}{n^2} \;=\; \frac{2\pi^3}{3}$$

and then to

$$\sum_{n \geq 1} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$. ///

[04.6] For fixed $y \in [0, 2\pi]$, show that there is *no* $f_y \in L^2[0, 2\pi]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0, 2\pi]$.

Discussion: Part of the issue here is whether L^2 functions truly have meaningful pointwise values at all, and we generally imagine that they do *not*, although such a negative fact may be hard to express formulaically.

Among many approaches, one is to suppose such f exists. Choose an orthonormal basis for $L^2[0, 2\pi]$ consisting of the continuous functions $\psi_n(x) = e^{2\pi inx}$, and see what the condition $\langle f_y, \psi_n \rangle = \psi_n(y)$ imposes on the alleged f_y. Indeed, this condition completely determines the Fourier coefficients of the alleged f_y, so

$$f_y \;=\; \sum_{n \in \mathbb{Z}} \psi_n(y) \cdot \psi_n \qquad \text{(with equality in an } L^2 \text{ sense)}$$

By Parseval,

$$|f_y|^2_{L^2} \;=\; \sum_n |\psi_n(y)|^2 \;=\; +\infty$$

since $|\psi_n(y)| = 1$ for all n. Thus, there can be no such f_y in L^2. ///

[04.7] (In contrast to the previous example's outcome.) Let V be the complex vector space of power series $f(z) = \sum_{n\geq 0} c_n\, z^n$ convergent on the open unit disk D in $\mathbb{C}$, having finite norm

$$|f| \;=\; \Big(\int_D |f(x+iy)|^2\; dx\,dy\Big)^{\frac{1}{2}}$$

with hermitian inner product

$$\langle f, g\rangle \;=\; \int_D f(x+iy)\cdot \overline{g(x+iy)}\; dx\,dy$$

Show that $\langle z^m, z^n\rangle = 0$ unless $m = n$, in which case it is $\frac{2\pi}{2n+1}$, and that $\psi_n(z) = z^n \cdot \frac{\sqrt{2n+1}}{\sqrt{2\pi}}$ is an orthonormal basis for V. Show that the sum $f_w(z) = \sum_{n\geq 0} \psi_n(z)\,\overline{\psi_n(w)}$ converges absolutely for $z, w \in D$, and that

$$\langle g(-),\, f_w\rangle \;=\; g(w) \qquad\qquad \text{(for } w \text{ in the disk)}$$

Show that for each fixed $w \in D$, pointwise evaluation $g \to g(w)$ is a continuous linear functional on V.
