(*November 16, 2016*)

Examples discussion 05

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-ex-05.pdf]

[05.1] Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) *closed*.

Discussion: Let $v_1, \ldots, v_m$ be an orthonormal basis for the given vector subspace W. For a Cauchy sequence $\{w_n\}$ in W, we claim that for each j the sequence $\langle w_n, v_j\rangle$ is Cauchy: by Cauchy-Schwarz-Bunyakowsky,

$$|\langle w_n, v_j\rangle - \langle w_{n'}, v_j\rangle| \;=\; |\langle w_n - w_{n'}, v_j\rangle| \;\le\; |w_n - w_{n'}|\cdot|v_j| \;=\; |w_n - w_{n'}|$$

Thus, by completeness of $\mathbb{R}$ and/or $\mathbb{C}$, that sequence has a limit c_j. As expected, we claim that $\lim_n w_n = \sum_{j=1}^m c_j \cdot v_j$. Indeed, using the orthonormality of the v_j's,

$$\Big|w_n - \sum_{j=1}^m c_j\cdot v_j\Big|^2 \;=\; \Big|w_n - \sum_{j=1}^m \lim_i \langle w_i, v_j\rangle\cdot v_j\Big|^2 \;=\; \Big|\sum_{j=1}^m \lim_i \langle w_n - w_i, v_j\rangle\cdot v_j\Big|^2$$

$$\le\; \sum_{j=1}^m |\lim_i \langle w_n - w_i, v_j\rangle|^2 \;=\; \lim_i \sum_{j=1}^m |\langle w_n - w_i, v_j\rangle|^2 \;\le\; \lim_i \sum_{j=1}^m |w_n - w_i|^2\cdot|v_j|^2 \;=\; \lim_i m\cdot|w_n - w_i|^2$$

Take n_o large enough so that $|w_n - w_i| < \varepsilon$ for $i, n \ge n_o$. Then the latter expression is at most $m\cdot\varepsilon$. This holds for all $\varepsilon > 0$, so the limit is 0. ///

[05.2] For a subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the closure of the subspace W in V.

Discussion: Let $\lambda_x(v) = \langle v, x\rangle$ for $x, v \in V$. Then $W^\perp = \bigcap_{w\in W} \ker \lambda_w$. Similarly, $(W^\perp)^\perp = \bigcap_{x\in W^\perp} \ker\lambda_x$. From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakowsky, each λ_x is continuous, so $\ker\lambda_x = \lambda_x^{-1}(\{0\})$ is closed, since $\{0\}$ is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so $(W^\perp)^\perp$ is closed.

Certainly $(W^\perp)^\perp \supset W$, because for each $w \in W$, $\langle x, w\rangle = 0$ for all $x \in W^\perp$. Thus, $(W^\perp)^\perp$ is a closed subspace, containing W. Being a closed subspace of a Hilbert space, $(W^\perp)^\perp$ is a Hilbert space itself. If $(W^\perp)^\perp$ were strictly larger than the topological closure $\overline{W}$ of W, then there would be $0 \ne y \in (W^\perp)^\perp$ orthogonal to $\overline{W}$. Then y would be orthogonal to W itself, so $0 \ne y \in W^\perp$, contradicting $0 \ne y \in (W^\perp)^\perp$. ///

[05.3] Let $T : \ell^2 \to \ell^2$ be the *right shift*: $T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$. Determine the *adjoint* T^*.

Discussion: The adjoint characterization is $\langle Tv, w\rangle = \langle v, T^*w\rangle$. That means that, for $(w_1, w_2, \ldots)$ in ℓ^2, we want

$$\langle (z_1, z_2, \ldots),\; T^*(w_1, w_2, \ldots)\rangle \;=\; \langle T(z_1, z_2, \ldots),\; (w_1, w_2, \ldots)\rangle \;=\; \langle (0, z_1, z_2, \ldots),\; (w_1, w_2, \ldots)\rangle$$

$$=\; z_1 w_2 + z_2 w_3 + z_3 w_4 + \ldots \;=\; \langle (z_1, z_2, \ldots),\; (w_2, w_3, \ldots)\rangle$$

Thus, we see that $T^*(w_1, w_2, w_3, \ldots) = (w_2, w_3, \ldots)$. That is, it is the *left* shift (yes, that loses the w_1-coordinate. ///

[05.4] Show that for $0 < x < 1$

$$\sum_{n\ge1} \frac{\sin 2\pi n x}{n} \;=\; \pi(\tfrac{1}{2} - x)$$

Discussion: The Fourier series of the right-hand side is computed to be that given on the left-hand side. By the Fourier-Dirichlet result on pointwise convergence, since $\pi(\frac{1}{2}-x)$ is finitely-piecewise C^o, and has left and right derivatives in $(0,1)$, its Fourier series converges to it pointwise there. ///

[05.5] Prove that every $f \in C^o_c(\mathbb{R})$ can be uniformly approximated (in sup norm) arbitrarily well as superpositions of Gaussians: given $\varepsilon > 0$, there is $\varphi \in C^o_c(\mathbb{R})$ and sufficiently large n such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi)\cdot ne^{-\pi n^2(\xi-x)^2}\,d\xi \Big| \;<\; \varepsilon$$

Discussion: This is an instance of an *approximate identity* and the basic property of such. Namely, for an approximate identity $\{\varphi_n\}$ on $\mathbb{R}$ and $f \in C^o_c(\mathbb{R})$, we have

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi)\cdot f(x+\xi)\,d\xi \Big| \;\longrightarrow\; 0 \qquad\qquad (\text{as } n \to +\infty)$$

By replacing ξ by $\xi - x$ in the integral, we have

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} f(\xi)\cdot \varphi(\xi - x)\,d\xi \Big| \;\longrightarrow\; 0 \qquad\qquad (\text{as } n \to +\infty)$$

Rather than reproving this general assertion in the example at hand, we simply clarify the interpretation in terms of approximate identities. That is, with $\varphi_1(x) = e^{-\pi x^2}$, we that the sequence $\varphi_n(x) = n\cdot\varphi_1(nx)$ is an approximate identity. More generally, we prove

[0.0.1] Claim: Let $\varphi \in C^o(\mathbb{R})$ be a non-negative $\mathbb{R}$-valued function, with $\int_{\mathbb{R}} \varphi = 1$. Then $\varphi_n(x) = n\cdot\varphi(n\cdot x)$ is an approximate identity.

Proof: The non-negative real-valued-ness is of course immediate. The integral of φ_n is

$$\int_{\mathbb{R}} \varphi_n(x)\,dx \;=\; \int_{\mathbb{R}} n\cdot\varphi(n\cdot x)\,dx \;=\; \int_{\mathbb{R}} n\cdot\varphi(x)\,\frac{dx}{n} \;=\; \int_{\mathbb{R}} \varphi(x)\,dx \;=\; 1$$

by replacing x by x/n in the integral. Finally, to see that the masses of the φ_n bunch up near 0: Since $\varphi \geq 0$ and

$$\lim_n \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x)\,dx \;=\; \int_{\mathbb{R}} \varphi(x)\,dx \;=\; 1$$

given $\varepsilon > 0$ there is sufficiently large n_o such that for all $n \geq n_o$

$$1 \;\leq\; \lim_n \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x)\,dx \;>\; 1-\varepsilon$$

Then, by replacing x by x/n in the integral,

$$\int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \varphi_n(x)\,dx \;=\; \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} n\cdot\varphi(n\cdot x)\,dx \;=\; \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x)\,dx \;>\; 1-\varepsilon$$

The verifies the bunching-up property. ///

[05.6] Without worrying too much about identifying the finite, positive constant $\int_{\mathbb{R}} \frac{(\sin x)^2}{x^2}\,dx$, prove that, for given $f \in C^o_c(\mathbb{R})$, given $\varepsilon > 0$, there is sufficiently large n and a function $\varphi \in C^o_c(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi)\cdot \frac{(\sin n(x-\xi))^2}{(x-\xi)^2}\,d\xi \Big| \;<\; \varepsilon$$

Discussion: After the more general discussion of the previous example, this is just another such. ///