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Examples 06

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[06.1] Let $c_1, c_2, \ldots$ be positive real, converging monotonically to 0. For $0 < x < 1$, prove that $\sum_{n\ge 0} c_n\, e^{2\pi i n x}$ *converges pointwise.*

Discussion: The expression as a Fourier series should not distract us from seeing an instance of the generalized alternating-decreasing criterion again, sometimes called *Dirichlet's criterion*: for a positive real sequence $c_1, c_2, \ldots$ monotone-decreasing to 0, and for a (possibly complex) sequence $b_1, b_2, \ldots$ with *bounded partial sums* $B_n = b_1 + \ldots + b_n$, the sum $\sum_n b_n c_n$ converges. The partial sums $\sum_{n\le N} e^{2\pi i n x}$ are bounded for $0 < x < 1$, by summing geometric series, so this criterion applies here.

The proof of the criterion itself is by *summation by parts*, a discrete analogue of integration by parts. That is, rewrite the tails of the sum as

$$\sum_{M\le n\le N} b_n c_n \;=\; \sum_{M\le n\le N} (B_n - B_{n-1}) c_n \;=\; -B_{M-1} c_M + \sum_{M\le n\le N} B_n (c_n - c_{n+1}) + B_N c_{N+1}$$

Since the partial sums are bounded, the first and last summand go to 0. Letting β be a bound for all the $|B_n|$, the summation is

$$\Big| \sum_{M\le n\le N} B_n (c_n - c_{n+1}) \Big| \;\le\; \sum_{M\le n\le N} |B_n| \cdot |c_n - c_{n+1}| \;=\; \sum_{M\le n\le N} |B_n| \cdot (c_n - c_{n+1}) \;\le\; \sum_{M\le n\le N} \beta \cdot (c_n - c_{n+1})$$

$$=\; \beta \cdot \sum_{M\le n\le N} (c_n - c_{n+1}) \;=\; \beta \cdot (c_M - c_{N+1})$$

by telescoping the series. Again, c_M and c_{N+1} go to 0. ///

[06.2] Show that the sup-norm completion of the space $C^o_c(\mathbb{R})$ of compactly-supported continuous functions is the space $C^o_o(\mathbb{R})$ of continuous functions going to 0 at infinity. An analogous assertion and argument should hold for any topological space in place of $\mathbb{R}$.

Discussion: The argument for this is general enough that we can replace $\mathbb{R}$ by a more general topological space X, probably locally compact and Hausdorff so that Urysohn's lemma assures us a good supply of continuous functions for auxiliary purposes. Then $C^o_o(X)$ is defined to be the collection of continuous functions f such that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

First, show that any $f \in C^o_o(\mathbb{R})$ is a sup-norm limit of functions from $C^o_c(\mathbb{R})$. Given $\varepsilon > 0$, let K be sufficiently large so that $|f(x)| < \varepsilon$ for $x \notin K$. We claim that there is an open $U \supset K$ with compact closure $\overline{U}$ (which would be obvious on $\mathbb{R}$ or $\mathbb{R}^n$). For each $x \in K$, let $U_x \ni x$ be an open set with compact closure (using the local compactness). By compactness of K, there is a finite subcover $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Then the closure of $U = U_{x_1} \cup \ldots \cup U_{x_n}$ is compact, as claimed. Then, invoking Urysohn's Lemma, let φ be a continuous function on X taking values in the interval $[0,1]$, that is 1 on K, and 0 off U, so φ has compact support. Then $\varphi \cdot f$ is continuous and has compact support, and

$$\sup_{x\in X} |f(x) - \varphi(x)\cdot f(x)| \;\le\; \sup_{x\in K} |f(x) - \varphi(x)\cdot f(x)| + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)| \;=\; 0 + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)|$$

$$\le\; \sup |1-\varphi| \cdot \sup_{x\notin K} |f(x)| \;<\; 1\cdot\varepsilon$$

That is, we can approximate f to within ε, as claimed.

On the other hand, now show that any sup-norm Cauchy sequence of $f_n \in C^o_c(X)$ has a pointwise limit f in $C^o_o(X)$. First, on any compact, the limit of the f_n's is *uniform* pointwise, so is continuous on compacts. Since every point $x \in X$ has a neighborhood U_x with compact closure, the pointwise limit is continuous on U_x. Thus, the pointwise limit is continuous at every point, hence continuous. Given $\varepsilon > 0$, take n_o sufficiently large so that $\sup_{x\in X} |f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \geq n_o$. Let K be the support of f_{n_o}. Then

$$\sup_{x\notin K} |f(x)| \;=\; \sup_{x\notin K} |f(x) - f_{n_o}(x)| \;\leq\; \sup_{x\in X} |f(x) - f_{n_o}| \;\leq\; \varepsilon$$

Thus, the pointwise limit goes to 0 at infinity. ///

[06.3] Show that the translation action $T_x f(y) = f(y+x)$ on the Banach space $C^o_{bdd}(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ is *not* continuous. That is, $\mathbb{R} \times C^o_{bdd}(\mathbb{R}) \to C^o_{bdd}(\mathbb{R})$ by $x \times f \to T_x f$ is *not* continuous. In particular, find a particular $f \in C^o_{bdd}(\mathbb{R})$ with $|f|_{C^o} = 1$ such that, there is a sequence $\delta_n \to 0$ of non-zero numbers δ_n such that $|T_{\delta_n} f - f|_{C^o} = 1$.

Discussion: The point is that on a non-compact topological space there may exist continuous, bounded, but *not uniformly continuous* functions, such as $f(x) = \sin(x^2)$. Let $x_n = n \cdot \sqrt{2\pi}$ and let $\delta_n > 0$ be a sequence of small positive reals going to 0 such that $(x_n + \delta_n)^2 = x_n^2 + \frac{\pi}{2}$. Then $\sin(x_n^2) = 0$, while $\sin((x_n+\delta_n)^2) = 1$, so the sup norm of $\sin(x^2) - \sin((x+\delta_n)^2)$ is 1. ///

[0.0.1] Remark: Nevertheless, the translation action *is* continuous on $C^o_o(\mathbb{R})$, which we see as follows. Given $f \in C^o_o(\mathbb{R})$, for given $\varepsilon > 0$, by a previous example there is $g \in C^o_c(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |g(x) - f(x)| < \varepsilon$. Since g is compactly supported, it is *uniformly* compact, so there is $\delta > 0$ such that $|x - y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Then for $|h| < \delta$,

$$\sup_{x\in\mathbb{R}} |f(x+h) - f(x)| \;\leq\; \sup_{x\in\mathbb{R}} |f(x+h) - g(x+h) - (f(x) - g(x))| + \sup_{x\in\mathbb{R}} |g(x+h) - g(x)| \;\leq\; \sup_{x\in\mathbb{R}} |f(x+h) - g(x+h)| + \sup_{x\in\mathbb{R}} |f(x) - g(x)| -$$

This is half the desired continuity, in contrast to the problem with $C^o_{bdd}(\mathbb{R})$. Similarly, the translation action $\mathbb{R} \times C^o_o(\mathbb{R}) \to C^o_o(\mathbb{R})$ is jointly continuous in both arguments.

[06.4] Prove that the *Volterra operator* $V f(x) = \displaystyle\int_0^x f(t)\, dt$ on $C^o[0,1]$ or on $L^2[0,1]$ has no (not-identically-zero) eigenvalues/eigenvectors.

Discussion: It suffices to consider $f \in L^2[0,1]$, since $C^o[0,1] \subset L^2[0,1]$. First consider $\lambda \neq 0$. The initial step is a sort of *bootstrapping* process to see that any eigenfunction $f \in L^2[0,1]$ would have to be in $C^1[0,1]$ (and, in fact, in $C^\infty[0,1]$). For $f \in L^2[0,1]$ and $0 \leq x < y \leq 1$, by Cauchy-Schwarz-Bunyakowsky,

$$|f(y) - f(x)| \;=\; \Big|\frac{1}{\lambda}\int_x^y f(t)\, dt\Big| \;\leq\; \frac{1}{|\lambda|}\Big(\int_x^y |f|^2\Big)^{\frac{1}{2}} \cdot \Big(\int_x^y 1\Big)^{\frac{1}{2}} \;\leq\; \frac{1}{|\lambda|}|f|_{L^2} \cdot |y - x|^{\frac{1}{2}}$$

giving continuity. For continuous f such that $\lambda \cdot f(x) = \int_0^x f(t)\, dt$, since the integrand is continuous, the integral is C^1 as a function of x, by the fundamental theorem of calculus. Differentiating both sides of the equation, $\lambda \cdot f'(x) = f(x)$ for all x. Also, the integral is 0 at $x = 0$, so $f(0) = 0$. We claim that the constant-coefficient differential equation $\lambda \cdot f' - f = 0$ with condition $f(0) = 0$ has only the zero solution. Indeed, all the solutions are of the form $f(x) = c \cdot e^{x/\lambda}$ for some constant c. (We can *prove* this widely-believed fact via the Mean Value Theorem: write a solution f as $f(x) = e^x \cdot g(x)$ for $g(x) = f(x)/e^x$. Then the differential equation becomes $\lambda(e^{x/\lambda} \cdot g)' - (e^x \cdot g) = 0$, which simplifies to $g' = 0$. The Mean Value Theorem assures us that g is constant.) Thus, $f(0) = 0$ implies $c = 0$, and f must be identically 0.

For $\lambda = 0$, the equation $0 = \int_0^x f(t)\, dt$ holds identically in x, so $\int_x^y f = 0$ for all $0 \leq x < y \leq 1$. That is, such $f \in L^2[0,1]$ is orthogonal to all characteristic functions of intervals. It is plausible that this implies that $f = 0$ (in the $L^2[0,1]$ sense). We can take advantage of the fact that we know that $1, \ldots \sin 2\pi n x, \cos 2\pi n x, \ldots$

for $n = 1, 2, 3, \ldots$ is an orthonormal basis for $L^2[0,1]$: if we can show that the L^2-closure of the span of characteristic functions of intervals contains all functions $\sin 2\pi nx, \cos 2\pi nx$, then f would be orthogonal to all these, hence 0. Indeed, the usual discussion of Riemann integrals of continuous functions is more than enough to show that continuous functions can be approximated arbitrarily well by linear combinations of characteristic functions of intervals. ///

[06.5] Let $K(x,y) = |x-y|$, and let

$$Tf(x) \;=\; \int_a^b K(x,y)\; f(y)\; dy \qquad\qquad (\text{for } f \in L^2[a,b])$$

Find some eigenvalues/eigenfunctions for the operator T. (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

Discussion: Take $\lambda \neq 0$. First, a bootstrapping procedure shows that a λ-eigenfunction f is at least C^1, as follows. From $\lambda \cdot f(x) = \int_a^b |x-y|\; f(y)\; dy$, with $0 \le x < x' \le 1$,

$$|f(x) - f(x')| \;\le\; \frac{1}{|\lambda|} \int_0^1 \Big| |x'-y| - |x-y| \Big| \cdot |f(y)\; dy$$

There are three cases: $0 \le y \le x$, $x < y < x'$, and $y \ge x'$. In the first and third, $||x'-y| - |x-y|| = |x'-x|$. In the second, we have uniform estimate

$$\Big| |x'-y| - |x-y| \Big| \;\le\; |x'-y| + |x-y| \;=\; \le\; 2|x'-x|$$

Then Cauchy-Schwarz-Bunyakowsky gives

$$|f(x) - f(x')| \;\le\; \frac{1}{|\lambda|} \cdot |f|_{L^2} \cdot 2|x'-x|$$

and f is continuous. Then, invoking the fundamental theorem of calculus, the eigenfunction property expressses f as a finite linear combination of C^1 functions:

$$\lambda \cdot f(x) \;=\; x \int_0^x f - \int_0^x y \cdot f(y)\; dy + \int_x^1 y \cdot f(y)\; dy - x \int_x^1 f$$

so f itself is C^1. Repeating, we find that f is C^2, justifying taking a second derivative of both sides of the equation (in a classical sense rather than distributional): first derivative is

$$\lambda \cdot f'(x) \;=\; \int_0^x f + xf(x) - xf(x) - xf(x) - \int_x^1 f + xf(x) \;=\; \int_0^x f - \int_x^1 f$$

and the second derivative is

$$\lambda \cdot f''(x) \;=\; f(x) + f(x) \;=\; 2f(x)$$

(For $\lambda \neq 0$) this gives the constant-coefficient equation $f'' = \frac{2}{\lambda} f$, which has solutions consisting of linear combinations of $x \to e^{x \cdot \sqrt{2/\lambda}}$.

With $f(x) = e^{cx}$, the integral transform can be evaluated explicitly by integrating breaking the integral into pieces and integrating by parts:

$$Tf(x) \;=\; \int_0^1 |x-y| \cdot e^{cy}\; dy \;=\; \int_0^x (x-y) \cdot e^{cy}\; dy - \int_x^1 (x-y) \cdot e^{cy}\; dy$$

$$=\; \Big[(x-y) \cdot \frac{e^{cy}}{c}\Big]_{y=0}^x - \int_0^x (-1) \cdot \frac{e^{cy}}{c}\; dy - \Big[(x-y) \cdot \frac{e^{cy}}{c}\Big]_{y=x}^1 + \int_x^1 (-1) \cdot \frac{e^{cy}}{c}\; dy$$

$$=\; \Big(0 - x\frac{1}{c}\Big) + \Big[\frac{e^{cy}}{c^2}\Big]_{y=0}^x - \Big((x-1)\frac{e^c}{c} - 0\Big) - \Big[\frac{e^{cy}}{c^2}\Big]_{y=x}^1$$

$$=\; -\frac{x}{c} + \Big(\frac{e^{cx}}{c^2} - \frac{1}{c^2}\Big) - (x-1)\frac{e^c}{c} - \Big(\frac{e^c}{c^2} - \frac{e^{cx}}{c^2}\Big) \;=\; 2\frac{e^{cx}}{c^2} - x\Big(\frac{1}{c} + \frac{e^c}{c}\Big) + \Big(-\frac{1}{c^2} + \frac{e^c}{c} - \frac{e^c}{c^2}\Big)$$

Such f cannot be an eigenfunction unless the linear terms vanish identically. We must examine the extent to which linear combinations $f(x) = Ae^{cx} + Be^{-cx}$ may cause the extra terms to cancel, under conditions on $c = \sqrt{2/\lambda}$: with such f,

$$T(Ae^{cx}+Be^{-cx}) \;=\; 2\frac{Ae^{cx}+B^{-cx}}{c^2} - x\Big(\frac{A}{c}+\frac{B}{-c}+\frac{Ae^c}{c}+\frac{Be^{-c}}{-c}\Big)+\Big(-\frac{A}{c^2}-\frac{B}{(-c)^2}+\frac{Ae^c}{c}+\frac{Be^{-c}}{-c}-\frac{Ae^c}{c^2}-\frac{Be^{-c}}{(-c)^2}\Big)$$

For the linear term to vanish identically, the coefficient of x and the constant coefficient must be 0. This gives a homogeneous system of two equations in the two unknowns A, B:

$$\begin{cases} \Big(\frac{1}{c}+\frac{e^c}{c}\Big)\cdot A & - & \Big(\frac{1}{c}+\frac{e^{-c}}{c}\Big)\cdot B & = & 0 \\ \Big(\frac{-1}{c^2}+\frac{e^c}{c}-\frac{e^c}{c^2}\Big)\cdot A & - & \Big(\frac{1}{c^2}+\frac{e^{-c}}{c}+\frac{e^{-c}}{c^2}\Big)\cdot B & = & 0 \end{cases}$$

This has a non-trivial solution if and only if the determinant is zero, multiplying through by c^3, this condition is

$$\begin{aligned} 0 \;&=\; -(1+e^c)\cdot(1+ce^{-c}+e^{-c}) + (-1+ce^c-e^c)\cdot(1+e^{-c}) \\ &=\; -(1+ce^{-c}+e^{-c}+e^c+c+1) + (-1+ce^c-e^c-e^{-c}+c-1) \\ &=\; -1-ce^{-c}-e^{-c}-e^c-c-1-1+ce^c-e^c-e^{-c}+c-1 \\ &=\; -4-ce^{-c}-2e^{-c}-2e^c+ce^c \;=\; -2(e^{c/2}+e^{-c/2})^2 + c(e^c-e^{-c}) \\ &=\; (e^{c/2}+e^{-c/2})\Big(c(e^{c/2}-e^{-c/2}) - 2(e^{c/2}+e^{-c/2})\Big) \end{aligned}$$

For $e^{c/2}+e^{-c/2}=0$, the first factor is 0, so $c/2 \in \frac{\pi i}{2}+\pi i\mathbb{Z}$ gives an eigenfunction. That is, $c \in \pi i + 2\pi i\mathbb{Z}$. Without worrying about possible zeros of the other factor of the determinant, taking $A = 1$, since the first equation in the system vanishes identically, we look at the second, obtaining

$$B \;=\; \frac{-1+ce^c-e^c}{1+ce^{-c}+e^{-c}} \;=\; \frac{ce^c - e^{c/2}(e^{c/2}+e^{-c/2})}{ce^{-c}+e^{-c/2}(e^{c/2}+e^{-c/2})} \;=\; \frac{ce^c}{ce^{-c}} \;=\; e^{2c} \;=\; 1 \qquad\qquad (\text{since } c\in\pi i+2\pi i\mathbb{Z})$$

Thus, for $c \in \pi i + 2\pi i\mathbb{Z}$,

$$e^{cx}+e^{-cx}$$

is an eigenfunction. That is, *at least* $\cos\pi x$, $\cos 3\pi x$, $\cos 5pix, \ldots$ are eigenfunctions. ///

[0.0.2] Remark: As might be suspected, there are more eigenvalues and eigenvectors, corresponding to zeros of the second factor in the determinant, but the eigenvalues λ and corresponding parameters $c = \sqrt{2/\lambda}$ are not as elementarily expressible as in the previous case. Indeed, with $c = ib$ and $b \in \mathbb{R}$, with $e^{c/2}+e^{-c/2} \neq 0$, vanishing of the second factor is

$$0 \;=\; ib(e^{ib/2}-e^{-ib/2}) - 2(e^{ib/2}+e^{-ib/2}) \;=\; b(e^{ib/2}+e^{-ib/2})\cdot\Big(-\frac{e^{ib/2}-e^{-ib/2}}{i(e^{ib/2}+e^{-ib/2})}-\frac{2}{b}\Big)$$

$$=\; b(e^{ib/2}+e^{-ib/2})\cdot\Big(-\tan\frac{b}{2}-\frac{2}{b}\Big)$$

Since $\tan\frac{b}{2}$ is periodic and goes from $-\infty$ to $+\infty$ in each period, it intersects the curve $(b, -2/b)$ in each period. In fact, since both are monotone, they intersect exactly once in each period. That is, there is (at least) another batch of eigenvalues at least as numerous (asymptotically) as the previous.

This can be anticipated on general principles, if we observe that any function f expressible as a superposition of functions $x \to |x-y|$ on $[0,1]$ satisfies the *boundary conditions* $f'(0)+f'(1)=0$ and $\int_0^1 f = f(0)+f(1)$, since $x \to |x-y|$ has those properties. That is, rather than having *no* conditions on a function on the circle

$\mathbb{R}/\mathbb{Z}$, giving eigenfunctions $e^{2\pi inx}$, with eigenvalues $(2\pi in)^2$ with multiplicity *two* (for $n \neq 0$), the eigenvalues get pushed farther from 0 by the implicit boundary conditions.

Variant: To see what happens without the complications entailed by restricting to a finite interval, we can consider

$$Tf(x) \;=\; \int_{\mathbb{R}} |x-y| \cdot f(y)\, dy \qquad\qquad (\text{for } f \in C_c^o(\mathbb{R}))$$

Certainly *something* is required for convergence, since $y \to |x-y|$ is not in $L^2(\mathbb{R})$. Again, this is

$$Tf(x) \;=\; \int_{y\le x} (x-y)\cdot f(y)\, dy - \int_{y\le x} (x-y)\cdot f(y)\, dy$$

$$=\; x\int_{y\le x} f(y)\, dy - \int_{y\le x} yf(y)\, dy - x\int_{y\ge x} f(y)\, dy + \int_{y\ge x} yf(y)\, dy$$

The continuity of f assures that both integrals are continuously differentiable, by the fundamental theorem of calculus. Thus,

$$(Tf)'(x) \;=\; \int_{y\le x} f(y)\, dy + xf(x) - xf(x) - \int_{y\ge x} f(y)\, dy + xf(x) - xf(x) \;=\; \int_{y\le x} f(y)\, dy - \int_{y\ge x} f(y)\, dy$$

and the second derivative is $2f(x)$. Although there are no eigenfunctions, this shows that, given $f \in C_c^o(\mathbb{R})$, we can solve the equation $u'' = f$ by $u(x) = \int_{\mathbb{R}} |x-y|\cdot f(y)\, dy$. ///

[06.6] Show that the *principal value* functional

$$f \;\longrightarrow\; PV\int_{\mathbb{R}} \frac{f(x)}{x}\, dx \;=\; \lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x}\, dx\Big)$$

is equal to

$$-\int_{\mathbb{R}} f'(x)\cdot \log|x|\, dx$$

for f *continuously* differentiable near 0, with sufficient hypotheses on the decay of f and f' at infinity. For example, suppose that $f \in L^2(\mathbb{R})$, $f(x) \to 0$ as $|x| \to \infty$, and $|f'(x)| \ll \frac{1}{1+x^2}$ for easy convergence.

Discussion: Careful integration by parts...
