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Examples Discussion 07

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-disc-07.pdf]

[07.1] Show that the direct sum $X \oplus Y$ of two Hilbert spaces (the underlying set is the collection of ordered pairs (x, y) with $x \in X$ and $y \in Y$, also denoted $x \oplus y$) with

$$\langle x \oplus y,\ x' \oplus y' \rangle \;=\; \langle x, x' \rangle_X + \langle y, y' \rangle_Y \qquad\qquad (\text{for } x, x' \in X \text{ and } y, y' \in Y)$$

is a Hilbert space.

Discussion: Checking that the indicated formula gives an inner product is straightforward. For example, additivity in the first argument:

$$\langle x_1 \oplus y_1 + x_2 \oplus y_2,\ x' \oplus y' \rangle \;=\; \langle (x_1 + x_2) \oplus (y_1 + y_2),\ x' \oplus y' \rangle_{X \oplus Y} \;=\; \langle x_1 + x_2, x' \rangle_X + \langle y_1 + y_2, y' \rangle_Y$$

$$= \Big(\langle x_1, x' \rangle_X + \langle x_2, x' \rangle_X \Big) + \Big(\langle y_1, y' \rangle_Y + \langle y_2, y' \rangle_Y \Big) \;=\; \langle x_1 \oplus y_1,\ x' \oplus y' \rangle + \langle x_2 \oplus y_2,\ x' \oplus y' \rangle$$

Completeness follows from the completeness of a cartesian product of two metric spaces. ///

[07.2] Show that the direct sum $X \oplus Y$ of two Banach spaces with

$$|x \oplus y| \;=\; |x|_X + |y|_Y \qquad\qquad (\text{for } x \in X \text{ and } y \in Y)$$

is a Banach space. Note that this produces a different outcome than the previous example when X, Y are Hilbert spaces. To encompass both examples at once, show that for any $p \geq 1$

$$|x \oplus y|_p \;=\; \Big(|x|_X^p + |y|_Y^p \Big)^{1/p} \qquad\qquad (\text{for } x \in X \text{ and } y \in Y)$$

makes $X \oplus Y$ a Banach space.

Discussion: The first part is similar to, and easier than, the previous example. The more general case further needs to invoke Minkowski's inequality. ///

[07.3] Let $\psi_n(x) = e^{2\pi i n x}$. Let $\delta_{\mathbb{Z}}$ be the *Dirac comb*, that is, a periodic version of Dirac's δ, describable as having Fourier series

$$\delta_{\mathbb{Z}} \;=\; \sum_{n \in \mathbb{Z}} 1 \cdot \psi_n \qquad\qquad (\text{converging in } H^{-1}(\mathbb{T}) \text{ or even } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon > 0)$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda \cdot u \;=\; \delta_{\mathbb{Z}}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division.* Show that the equation $v'' - \lambda v = f$ is solved by

$$v(x) \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_0^1 u(x-t)\, f(t)\, dt$$

Discussion: Using the *spectral* characterization of the $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ norm,

$$\Big| \sum_{n \in \mathbb{Z}} 1 \cdot \psi_n \Big|^2_{H^{-\frac{1}{2}-\varepsilon}} \;=\; \sum_{n \in \mathbb{Z}} |1|^2 \cdot (1 + n^2)^{-\frac{1}{2}-\varepsilon}$$

which is convergent for all $\varepsilon > 0$, by comparison to $\sum_{n\neq 0} 1/n^2$. So that Fourier series converges in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ and produces a *generalized function* there.

The extension by continuity of d/dx from $C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ to $\widetilde{\frac{d}{dx}} : H^s(\mathbb{T}) \to H^{s-1}(\mathbb{T})$ is continuous, by design. Similarly, $\widetilde{\frac{d^2}{dx^2}} : H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ is continuous. That is, since infinite sums are the corresponding limits of finite partial sums, this continuity means that termwise differentiation is correct. Let $u = \sum_n c_n \psi_n$, and solve, dropping the tilde from the notation,

$$\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \;=\; \delta_{\mathbb{Z}} \;=\; u'' - \lambda u \;=\; \sum_n c_n \Big(\frac{d^2}{dx^2} - \lambda\Big)\psi_n \;=\; \sum_n c_n(-4\pi^2 n^2 - \lambda) \cdot \psi_n$$

Equating coefficients, $c_n = 1/(-4\pi^2 n^2 - \lambda)$, for λ not equal to $-4\pi^2 n^2$ for integer n. Another easy estimate shows that this u has gained 2 Sobolev indices, so is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{T})$.

By Sobolev imbedding/inequality, $H^s(\mathbb{T}) \subset C^o(\mathbb{T})$ for all $s > \frac{1}{2}$, so the solution is continuous (and, in fact, satisfies a further Lipschitz condition).

To see that $v'' - \lambda v = f$ is solved by

$$v(x) \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_0^1 u(x-t)\, f(t)\, dt$$

take f such that $\widehat{f} \in \ell^1_{\mathbb{Z}}(\mathbb{T})$, meaning that $\sum_n |\widehat{f}(n)| < \infty$. A somewhat stronger, more intuitive assumption is that $f \in C^2(\mathbb{T})$, and then by integration by parts

$$\widehat{f''}(n) \;=\; \int_{\mathbb{T}} e^{-2\pi i n x}\, f''(x)\, dx \;=\; \int_{\mathbb{T}} (-2\pi i n)^2\, e^{-2\pi i n x}\, f(x)\, dx \;=\; (2\pi i n)^2 \cdot \widehat{f}(n)$$

(On the circle $\mathbb{T}$, and/or for $\mathbb{Z}$-periodic functions, there are no boundary terms in integration by parts.) We do not even to invoke Riemann-Lebesgue, since $|\widehat{f''}(n)|$ is *bounded*, so there is a constant C such that $|\widehat{f}(n)| \le C/n^2$, so $\widehat{f} \in \ell^1_{\mathbb{Z}}$.

Then Fubini-Tonelli assures the legitimacy of interchanging sum and limit: [1]

$$\int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_{\mathbb{T}} \sum_m \frac{1}{-4\pi^2 m^2 - \lambda} e^{2\pi i m(x-t)} \cdot \sum_n \widehat{f}(n) e^{2\pi i n t}\, dt$$

$$= \sum_{m,n} \frac{\widehat{f}(n)}{-4\pi^2 m^2 - \lambda} \int_{\mathbb{T}} e^{2\pi i m(x-t)}\, e^{2\pi i n t}\, dt \;=\; \sum_n \frac{\widehat{f}(n)}{-4\pi^2 n^2 - \lambda} e^{2\pi i n x}$$

by mutual orthogonality of distinct exponentials. By Riemann-Lebesgue, $\widehat{f}(n) \to 0$, so

$$\sum_n |\widehat{f}(n)|^2 \cdot (1+n^2)^s \;<\; \infty \qquad\qquad (\text{for any } s < -\tfrac{1}{2})$$

so $f \in H^{-1}(\mathbb{T})$, for example. Application of the (extension of) $\frac{d^2}{dx^2} - \lambda$ *termwise* (again, justified by continuity of the extension) produces the Fourier expansion of f. ///

[0.1] Remark: The previous example illustrates the utility of using *generalized functions* even in a discussion that seems not to refer to them: there was no need to *guess* the function $u(x-t)$ (sometimes called a *Green's*

[1] In fact, we will see later that for u continuous and f in any Sobolev space, the interchange is justified.

function) solving the differential equation, since we *solved* for it using the Fourier expansion of $\delta_{\mathbb{Z}}$ that only converges in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$.

[07.4] Let V be a vector space, with norms $|\cdot|_1$ and $|\cdot|_2$. Suppose that $|v|_2 \geq |v|_1$ for all $v \in V$. Show that the identity map $i : V \to V$ is continuous, where the source is given the $|\cdot|_2$ topology and the target is given the $|\cdot|_1$ topology. Show that if a sequence $\{v_n\}$ in V is $|\cdot|_2$ Cauchy, then it is $|\cdot|_1$-Cauchy. Let V_j be the completion of V with respect to the metric $|v - v'|_j$. Show that we can *extend i by continuity* to a continuous linear map $I : V_2 \to V_1$, that is, by

$$I(V_2\text{-limit of } V_2\text{-Cauchy sequence } \{v_n\}) \;=\; V_1\text{-limit of } \{v_n\}$$

Discussion: First, it suffices to show that the identity map $i : V \to V$ with indicated topologies is *bounded*, and, indeed,

$$|j(v)|_1 \;=\; |v|_1 \;\leq\; |v|_2 \qquad\qquad (\text{for all } v \in V, \text{ by hypothesis})$$

For $\{v_n\}$ Cauchy in the $|\cdot|_2$ topology, given $\varepsilon > 0$, take n_o such that $|v_m - v_n|_2 < \varepsilon$ for all $m, n \geq n_o$. Then the same inequality holds (with the same n_o and ε) for $|\cdot|_2$, so $\{v_n\}$ is Cauchy in the $|\cdot|_1$ topology.

A useful characterization of the completion $\widetilde{X}$ of a metric space X is that there is an isometry $j : X \to \widetilde{X}$, and any non-expanding[2] map $f : X \to Y$ to a complete metric space Y extends uniquely to continuous map $\widetilde{f} : \widetilde{X} \to Y$, with $\widetilde{f} \circ j = f$. In particular,

$$\widetilde{f}(X - \lim_n x_n) \;=\; Y - \lim_n f(x_n)$$

This is well-defined because f is continuous on X. Thus, with $X = V$, $Y = V^2$, and $f : V \to V_2$ given by inclusion, we have the assertion. ///

[07.5] Let X, Y be Hilbert spaces over $\mathbb{R}$, to avoid the distraction of complex conjugation. Let $i_X : X \to X^*$ be the Riesz-Fréchet isomorphism of X to its dual X^*, by $x \to \langle -, x\rangle$. For a continuous linear $T : X \to Y$, let $T^* : Y^* \to X^*$ be the *adjoint*, defined as usual by $(T^*\mu)(x) = \mu(Tx)$ for $\mu \in Y^*$ and $x \in X$. Perhaps surprisingly, the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X^* & \xleftarrow{T^*} & Y^* \end{array}$$

does not generally *commute*. Give an example of failure. (*Hint:* It already suffices to take $X = Y = \mathbb{R}$. In fact, the diagram commutes *if and only* if $T : X \to Y$ is an *isometry* to the image $T(X)$.)

[07.6] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2 \, dx$. (*Hint:* use Plancherel.)

Discussion: From a standard stock of easy Fourier transforms, the Fourier transform of a characteristic function of a symmetrical interval is very close to the given function:

$$\widehat{\mathrm{ch}_{[-1,1]}}(\xi) \;=\; \int_{-1}^{1} e^{-2\pi i \xi x}\, dx \;=\; \frac{e^{-2\pi i\xi} - e^{2\pi i \xi}}{-2\pi i \xi} \;=\; \frac{\sin 2\pi\xi}{\pi\xi}$$

Applying Plancherel, we have

$$2 \;=\; \int_{\mathbb{R}} |\mathrm{ch}_{[-1,1]}|^2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin 2\pi\xi}{\pi\xi}\Big)^2 \, d\xi$$

[2] This sense of *non-expanding* is the reasonable one: $d_Y(f(x), f(x')) \leq d_X(x, x')$ for all $x, x' \in X$.

The change of variables replacing ξ by $\xi/2\pi$ gives

$$2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi/2}\Big)^2 \,\frac{d\xi}{2\pi} \;=\; \frac{2}{\pi}\int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi}\Big)^2 \,d\xi$$

Thus, the desired integral is π. ///

[07.7] Show that for any $f \in C_c^o(\mathbb{R})$, $\lim_{|t|\to+\infty} \int_{\mathbb{R}} e^{itx}\, f(x)\, dx = 0$.

Discussion: This is an instance of Riemann-Hilbert: it is for $L^1(\mathbb{R})$. Compactly supported continuous functions are in $L^1(\mathbb{R})$, so $\widehat{f}(\xi)$ is continuous and goes to zero at infinity. ///
