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Examples discussion 08

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[08.1] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\,dx \Big| \;<\; C \cdot \sqrt{\delta}$$

Formulate and prove the corresponding assertion for L^p with $1 < p < \infty$.

Discussion: Let h_δ be the characteristic function of $[t-\delta, t+\delta]$. By Cauchy-Schwarz-Bunyakowsky

$$\Big| \int_{t-\delta}^{t+\delta} f \Big| \;=\; |\langle f, h_\delta \rangle_{L^2}| \;\le\; |f|_{L^2} \cdot |h_\delta|_{L^2} \;=\; |f|_{L^2} \cdot \sqrt{2\delta}$$

The case of conjugate exponents $\frac{1}{p} + \frac{1}{q} = 1$ is the same, using Hölder's inequality rather than Cauchy-Schwarz-Bunyakowsky. There is no immediate analogue for L^1, although a weaker result is possible, as in the next example. ///

[08.2] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\,dx \Big| \;<\; \varepsilon$$

Sharpen the first example to show that

$$\int_{t-\delta}^{t+\delta} f(x)\,dx \;=\; o(\sqrt{\delta}) \qquad\qquad (\text{as } \delta \to 0^+)$$

where Landau's little-o notation is that $f(x) = o(g(x))$ as $x \to a$ when $\lim_{x\to a} f(x)/g(x) = 0$.

Discussion: Let $S_n = \{x : \frac{1}{n+1} \le |x-t| < \frac{1}{n}\}$. Then

$$\Big| \sum_{n\ge 1} \int_{S_n} f \Big| \;\le\; \sum_{n\ge 1} \int_{S_n} |f| \;\le\; |f|_{L^1}$$

Thus, the sum of non-negative terms $\sum_{n\ge1} \int_{S_n} |f|$ is convergent, so the tails $\sum_{n\ge N} \int_{S_n} |f|$ go to 0 as $N \to +\infty$. Thus,

$$\Big| \int_{|x-t|\le N} f \Big| \;\le\; \int_{|x-t|\le N} |f| \;=\; \sum_{n\ge N} \int_{S_n} |f|$$

goes to 0 as $N \to +\infty$. Then this idea can be applied to $\int_{|x-t|<\delta} |f|^p$ in the previous example. ///

[08.3] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you assert: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$.)

Discussion: The idea is to invoke $f * g = (\widehat{f} \cdot \widehat{g})^\wedge$ for *even* functions $f, g \in L^1$, since for even functions the inverse Fourier transform is the same as the forward Fourier transform. Conveniently, Gaussians are in $L^1 \cap L^2$, and have Fourier transforms which are again Gaussians:

$$\widehat{e^{-\pi a x^2}}(\xi) \;=\; \frac{1}{\sqrt{a}} e^{-\pi \xi^2/a} \qquad\qquad (\text{for } a > 0)$$

so

$$e^{-\pi x^2} * e^{-\pi x^2}(\xi) \;=\; \widehat{e^{-\pi x^2} \cdot e^{-\pi x^2}}(\xi) \;=\; \widehat{e^{-2\pi x^2}}(\xi) \;=\; \frac{1}{\sqrt{2}} e^{-\pi \xi^2/2}$$

For the other example, the bound $|f * g|_{L^1} \le |f|_{L^p} \cdot |g|_{L^q}$ for conjugate exponents p, q shows that $f * g \in L^1$ for $f, g \in L^2$. Thus, the same identity holds for $f, g \in L^2$, with the Plancherel extension of Fourier transform. That is, $\widehat{f}$ and $\widehat{g}$ need not be the literal integrals for the Fourier transform, but its extension by continuity to L^2. With χ_a the characteristic function of $[-a, a]$, recall that

$$\widehat{\chi_a}(\xi) \;=\; \int_{-a}^{a} e^{-2\pi i \xi x}\, dx \;=\; \Big[\frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\Big]_{-a}^{a} \;=\; \frac{e^{-2\pi i a \xi} - e^{2\pi i a \xi}}{-2\pi i \xi} \;=\; \frac{\sin 2\pi a \xi}{\pi \xi}$$

Thus,

$$(\pi \cdot \chi_{a/2\pi})\widehat{\ }(\xi) \;=\; \frac{\sin \xi}{\xi}$$

Then

$$\Big(\frac{\sin x}{x} * \frac{\sin x}{x}\Big)(\xi) \;=\; \Big((\pi \cdot \chi_{a/2\pi}) \cdot (\pi \cdot \chi_{a/2\pi})\Big)\widehat{\ }(\xi) \;=\; \pi \cdot (\pi \cdot \chi_{a/2\pi})\widehat{\ }(\xi) \;=\; \pi \cdot \frac{\sin \xi}{\xi}$$

[08.4] Let $K(x, y) \in L^2([a, b] \times [a, b])$, and attempt to define a map $T : L^2[a, b] \to L^2[a, b]$ by

$$Tf(x) \;=\; \int_a^b K(x, y)\, f(y)\, dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$|Tf|_{L^2[a,b]} \;\le\; |K|_{L^2([a,b]\times[a,b])} \cdot |f|_{L^2[a,b]}$$

(One would say that $K(,)$ is a *Schwartz kernel* for the map T. Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y, x)}$.

Discussion: By Fubini-Tonelli, $y \to K(x, y)$ is measurable for almost all x, so $Tf(x)$ is defined almost everywhere (assuming convergence of the integral). By Cauchy-Schwarz-Bunyakowsky, and Fubini-Tonelli as needed,

$$\int_a^b |Tf(x)|^2\, dx \;=\; \int_a^b \Big|\int_a^b K(x, y)\, f(y)\, dy\Big|^2\, dx \;\le\; \int_a^b \int_a^b |K(x, y)|^2\, dy \cdot \int_a^b |f(y')|^2\, dy'\, dx$$

$$=\; |f|_{L^2}^2 \cdot \int_a^b \int_a^b |K(x, y)|^2\, dx\, dy \;=\; |f|_{L^2[a,b]}^2 \cdot |K|_{L^2([a,b]\times[a,b])}^2 \;<\; +\infty$$

Thus, T is *bounded*, so is a *continuous* linear map of $L^2[a, b]$ to itself. ///

[08.5] The Volterra operator $Vf(x) = \int_0^x f(y)\, dy$ on $L^2[0, 1]$ has kernel

$$K(x, y) \;=\; \begin{cases} 1 & (\text{for } 0 \le y \le x \le 1) \\ 0 & (\text{for } 0 \le x < y \le 1) \end{cases}$$

Determine the (Schwartz) kernel for $T = V \circ V^*$. Find some eigenfunctions for T. (Recall that V has no eigenfunctions!) (*Hint:* apply d/dx to the equation $Tf = \lambda \cdot f$ and presume that the differentiation passes inside the integral.)

Discussion: The kernel $K^*(x,y)$ for an adjoint V^* is always obtained by the procedure $\overline{K(y,x)}$. Here, the kernel is real-valued. For two operators S,T with respective kernels $K_S(x,y)$ and $K_T(x,y)$, using Fubini-Tonelli as necessary,

$$(S\circ T)f(x) \;=\; (S(Tf))(x) \;=\; \int_0^1 K_S(x,t)\,Tf(t)\,dt \;=\; \int_0^1 K_S(x,t)\Big(\int_0^1 K_T(t,y)\,f(y)\,dy\Big)\,dt$$

$$= \int_0^1 \Big(\int_0^1 K_S(x,t)\,K_T(t,y)\,dt\Big)\,f(y)\,dy$$

so the kernel of the composite is

$$K_{S\circ T}(x,y) \;=\; \int_0^1 K_S(x,t)\,K_T(t,y)\,dt$$

Thus, the kernel $L(x,y)$ for $V\circ V^*$ is

$$K_{V\circ V^*}(x,y) \;=\; \int_0^1 K_V(x,t)\,K_V(y,t)\,dt \;=\; \int_0^1 \begin{cases} 1 & (\text{for } 0\le t\le x\le 1) \\ 0 & (\text{for } 0\le x< t\le 1)\end{cases} \cdot \begin{cases} 1 & (\text{for } 0\le t\le y\le 1) \\ 0 & (\text{for } 0\le y< t\le 1)\end{cases}\,dt$$

$$= \int_{t\le x,\; t\le y} 1\,dt \;=\; \min(x,y)$$

$$\int_0^1 \min(x,y)\,f(y)\,dy \;=\; \int_0^x y\,f(y)\,dy + \int_x^1 x\,f(y)\,dy \;=\; \int_0^x y\,f(y)\,dy + x\int_x^1 f(y)\,dy$$

For $f\in L^2$, Cauchy-Schwarz-Bunyakowsky shows that the latter expression side is *continuous* as a function of x. Thus, an eigenfunction equation $\lambda\cdot f = (V\circ V^*)f$ for $f\in L^2$ and $\lambda\neq 0$ implies that f is *continuous*. Then from

$$\lambda\cdot f(x) \;=\; \int_0^x y\,f(y)\,dy + x\int_x^1 f(y)\,dy$$

the fundamental theorem of calculus implies that $f\in C^1$. By induction on k, $f\in C^k$ for all k, so f is *smooth*. Differentiating the latter expression,

$$\lambda\cdot f'(x) \;=\; x\cdot f(x) + \int_x^1 f(y)\,dy - x\cdot f(x) \;=\; \int_x^1 f(y)\,dy$$

Differentiating again, $\lambda\cdot f'' = -f$. Thus, $f(x) = Ae^{cx} + Be^{-cx}$ for some constants A,B, with $c=\sqrt{-\lambda}$. But this is only a *necessary* condition for an eigenfunction, not *sufficient*.

One way to determine allowable λ is to directly compute the integral

$$\int_0^1 \min(x,y)\cdot(Ae^{cy}+Be^{-cy})\,dy$$

and examine the condition that this be equal to $-1/c^2\cdot(Ae^{cx}+Be^{-cx})$. This would involve two integrations by parts. Equivalently, but somewhat more lightly, for each fixed $x\in[0,1]$,

$$F(y) \;=\; \begin{cases} 0 & (\text{for } y<0) \\ y & (\text{for } 0\le y<x) \\ x & (\text{for } x\le y<1) \\ 0 & (\text{for } y\ge 1)\end{cases}$$

Integrating by parts twice *distributionally* gives

$$\int_0^1 \min(x,y)\cdot(Ae^{cy}+Be^{-cy})\,dy \;=\; \int_{\mathbb{R}} F(y)\cdot(Ae^{cy}+Be^{-cy})\,dy \;=\; \int_{\mathbb{R}} F''(y)\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy$$

$$= \int_{\mathbb{R}} \Big(\Big(\begin{cases} 0 & (\text{for } y<0) \\ 1 & (\text{for } 0\le y<x) \\ 0 & (\text{for } y\ge x)\end{cases}\Big) - x\cdot\delta_1\Big)'\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy$$

$$= \int_{\mathbb{R}}\Big(\delta_0-\delta_x-x\cdot\delta_1'\Big)\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy \;=\; \frac{A+B}{c^2}-\frac{Ae^{cx}+Be^{-cx}}{c^2}-x\cdot\frac{Ace^c-Bce^{-c}}{c^2}$$

For this to be $-(Ae^{cx}+Be^{-cx})/c^2$, it is necessary and sufficient that the two extra terms vanish, that is,

$$\frac{A+B}{c^2}-x\cdot\frac{Ace^c-Bce^{-c}}{c^2} \;=\; 0$$

for almost all $x\in[0,1]$. This is a linear function in x, so $A+B=0$ and then $e^c+e^{-c}=0$, which is $e^{2c}=-1$. That is, $c\in\pi i+2\pi i\mathbb{Z}$ with corresponding eigenfunction $e^{cx}-e^{-cx}$. ///

[08.6] The sawtooth function is first defined on $[0,1)$ by $\sigma(x)=x-\frac{1}{2}$, and then extended to $\mathbb{R}$ by periodicity so that $\sigma(x+n)=\sigma(x)$ for all $x\in\mathbb{R}$ and $n\in\mathbb{Z}$. Determine its Fourier expansion. Describe the derivative σ' of σ.

Discussion: The 0^{th} Fourier coefficient is 0. For $n\neq 0$, integrating by parts once, the n^{th} Fourier is $-1/2\pi in$. That is, at least converging in L^2,

$$\sigma(x) \;=\; \sum_{n\neq 0}\frac{1}{-2\pi in}\,e^{2\pi inx}$$

In fact, from the Fourier-coefficient criterion for Sobolev spaces, $\sigma\in H^{\frac{1}{2}-\varepsilon}$ for all $\varepsilon>0$. Differentiating termwise (in an extended sense),

$$\sigma' \;=\; -\sum_{n\neq 0} e^{2\pi inx} \qquad\qquad (\text{convergent in } H^{-\frac{1}{2}-\varepsilon})$$

We might recognize this as being closely related to the Dirac comb

$$\delta_{\mathbb{Z}} \;=\; \sum_{n\in\mathbb{Z}} e^{2\pi inx} \qquad\qquad (\text{convergent in } H^{-\frac{1}{2}-\varepsilon})$$

Specifically, $\sigma'=1-\delta_{\mathbb{Z}}$. Also, looking at the description of σ directly, its derivative is (locally) 1 away from $\mathbb{Z}$, and has a $-\delta_n$ for all $n\in\mathbb{Z}$. That is, yet again,

$$\sigma' \;=\; 1-\sum_{n\in\mathbb{Z}}\delta_n \;=\; 1-\delta_{\mathbb{Z}}$$

[08.7] Given f in the Schwartz space $\mathscr{S}$, show that there is $F\in\mathscr{S}$ with $F'=f$ if and only if $\int_{\mathbb{R}} f=0$.

Discussion: On one hand, if $f=F'$ for $F\in\mathscr{S}$, then $\int_{-\infty}^x f(y)\,dy=F(x)$. Since $\lim_{x\to+\infty}F(x)=0$, $\int_{\mathbb{R}} f=0$.

On the other hand, if $\int_{\mathbb{R}} f=0$, let $F(x)=\int_{-\infty}^x f$, and show that $F\in\mathscr{S}$. Since $F'=f$ by the fundamental theorem of calculus, the (higher) derivatives of F are those of f, so all that needs to be shown is that F itself is of rapid decay. For $x\to-\infty$,

$$|F(x)| \;\le\; \int_{-\infty}^x |f| \;\le\; \int_{-\infty}^x |y|^{-N}\cdot\sup_{t\in\mathbb{R}}|t^N\cdot f(t)|\,dy \;\le\; \sup_{t\in\mathbb{R}}|t^N\cdot f(t)|\cdot\int_{-\infty}^x |y|^{-N}\,dy \;=\; \sup_{t\in\mathbb{R}}|t^N\cdot f(t)|\cdot\frac{|x|^{1-N}}{N-1}$$

Using the condition $\int_{\mathbb{R}} f = 0$,

$$F(x) \;=\; \int_{-\infty}^{x} f \;=\; \int_{\mathbb{R}} f - \int_{x}^{\infty} f \;=\; 0 - \int_{x}^{\infty} f$$

so for $x \to +\infty$ it suffices to similarly estimate

$$\Big|\int_{x}^{\infty} f\Big| \;\le\; \int_{x}^{\infty} y^{-N} \cdot \sup_{t\in\mathbb{R}} |t^N \cdot f(t)|\; dy \;\le\; \sup_{t\in\mathbb{R}} |t^N \cdot f(t)| \cdot \int_{x}^{\infty} y^{-N}\; dy \;=\; \sup_{t\in\mathbb{R}} |t^N \cdot f(t)| \cdot \frac{x^{1-N}}{N-1}$$

giving the rapid decay. ///

[08.8] Show that the evaluation functional (Dirac delta) $\delta_{x_o} : f \to f(x_o)$ is a continuous linear functional on $\mathscr{S}$, for every $x_o \in \mathbb{R}$. Determine the Fourier transform of δ_{x_o}.

Discussion: The continuity uses just one of the seminorms giving the topology on $\mathscr{S}$: for $f, g \in \mathscr{S}$,

$$|\delta_{x_o}(f) - \delta_{x_o}(g)| \;\le\; \sup_{x\in\mathbb{R}} |(f-g)(x)|$$

Thus, given $\varepsilon > 0$, for $\sup_{x\in\mathbb{R}} |(f-g)(x)| < \delta = \varepsilon$, we have $|\delta_{x_o}(f) - \delta_{x_o}(g)| < \varepsilon$.

To compute the Fourier transform of δ_{x_o} it is entirely reasonable to imagine that the following produces the correct outcome:

$$\widehat{\delta_{x_o}}(\xi) \;=\; \int_{\mathbb{R}} \delta_{x_o}(\xi)\, e^{-2\pi i \xi x}\; dx \;=\; \delta_{x_o}(x \to e^{-2\pi i \xi x}) \;=\; e^{-2\pi i \xi x_o}$$

And, indeed, this conclusion is correct, and this is the effective way to think about and perform such computations. However, we do also want to understand why and how such a computation can be understood as being entirely rigorous. For example, the indicated integral cannot be a literal integral, since δ_{x_o} is not a pointwise-valued (classical) function. Likewise, although $\delta_{x_o} \in \mathscr{S}^*$, the exponential function is not in $\mathscr{S}$, but only in $\mathcal{E} = C^\infty(\mathbb{R})$. Using the duality characterization of $\mathcal{E}^*$, one can readily check that $\delta_{x_o} \in \mathcal{E}^*$, by letting K be any compact containing x_o, and then as for continuity on $\mathscr{S}$,

$$|\delta_{x_o}(f) - \delta_{x_o}(g)| \;\le\; \sup_{x\in K} |(f-g)(x)|$$

But it might be better to check that, viewing the apparent integral as an extension-by-continuity of literal integrals, with test functions (or Schwartz functions) f_n approaching δ_{x_o} in the $\mathscr{S}^*$ topology,

$$(\mathscr{S}^*\text{–}\lim_n f_n)\widehat{\;} \;=\; \mathscr{S}^*\text{–}\lim_n \widehat{f}_n \;=\; \mathscr{S}^*\text{–}\lim_n \int_{\mathbb{R}} f_n(x)\, e^{-2\pi i \xi x}\; dx \;=\; e^{-2\pi i \xi x_o}$$

The $\mathscr{S}^*$-continuity of the Fourier transform assures that the outcome is independent of the choice of $\{f_n\}$. Let φ be any test function that is non-negative real-valued and $\int_{\mathbb{R}} \varphi = 1$. Let $f_n(x) = \frac{1}{n}\varphi(n(x - x_o))$. This is a sort of *approximate identity* concentrated at x_o. By continuity of $x \to e^{-2\pi i \xi x}$ (and monotone convergence or less), we have the natural *pointwise estimate*

$$\lim_n \int_{\mathbb{R}} f_n(x)\, e^{-2\pi i \xi x} \;=\; e^{-2\pi i \xi x_o} \qquad\qquad \text{(for fixed } \xi \in \mathbb{R})$$

But this pointwise limit is not quite the $\mathscr{S}^*$ limit we want. Rather, we want to show that, for every $\eta \in \mathscr{S}$,

$$\lim_n \int_{\mathbb{R}} \eta(\xi) \cdot \Big(\int_{\mathbb{R}} f_n(x)\, e^{-2\pi i \xi x}\; dx \Big)\; d\xi \;=\; \int_{\mathbb{R}} \eta(\xi) \cdot e^{-2\pi i \xi x_o}\; d\xi$$

And, unsurprisingly, via Fubini-Tonelli,

$$\int_{\mathbb{R}} \eta(\xi) \cdot \Big(\int_{\mathbb{R}} f_n(x)\, e^{-2\pi i \xi x}\, dx \Big)\, d\xi \;=\; \int_{\mathbb{R}} f_n(x) \int_{\mathbb{R}} \eta(\xi)\, e^{-2\pi i \xi x}\, d\xi\, dx \;=\; \int_{\mathbb{R}} f_n(x)\, \widehat{\eta}(x)\, dx \;\longrightarrow\; \widehat{\eta}(x_o)$$

by the continuity of $\widehat{\eta} \in \mathscr{S}$.

In fact, the same argument justifies, for once and for all, the computation of Fourier transforms of $u \in \mathcal{E}^* \subset \mathscr{S}^*$ by $\widehat{u}(\xi) = u(x \to e^{-2\pi i \xi x})$. ///

[08.9] Let $u(x) = e^x \cdot \sin(e^x)$. Explain in what sense the integral $\int_{\mathbb{R}} f(x)\, u(x)\, dx$ converges for every $f \in \mathscr{S}$.

Discussion: The idea is to integrate by parts, noting that $u = v'$ with $v(x) = \cos(e^x)$. We must be careful with the boundary terms:

$$\int_{\mathbb{R}} f(x)\, u(x)\, dx \;=\; \int_{\mathbb{R}} f(x)\, v'(x)\, dx \;=\; \lim_{M,N\to+\infty} \int_{-M}^{N} f(x)\, v'(x)\, dx$$

$$=\; \lim_{M,N\to+\infty} \Big(\Big[f(x)\, v(x) \Big]_{-M}^{N} - \int_{-M}^{N} f'(x)\, v(x)\, dx \Big)$$

Since $v(x)$ is bounded and f' is of rapid decay, the limit *exists*, so the original integral is convergent. Further, the value is correctly determined by integration by parts, namely

$$-\int_{-\infty}^{\infty} f'(x)\, v(x)\, dx \;=\; -\int_{-\infty}^{\infty} f'(x)\, \cos(e^x)\, dx$$

That is, for $f \in \mathscr{S}$ and functions such as u obtained by differentiating bounded smooth functions, integration by parts is completely justifiable via the natural estimates. ///