(*February 14, 2017*)

Examples discussion 09

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[09.1] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this kind of construction to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0,1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1,2,3,\ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n,n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1), -n]$ and monotonically from 1 to 0 in $[n, n+1]$.

Discussion: In $x > 0$, by induction, the derivatives are finite linear linear combinations of functions of the form $x^{-n}e^{-1/x}$. It suffices to show that $\lim_{x\to 0^+} x^{-n}e^{-1/x} = 0$. Equivalently, that $\lim_{x\to+\infty} x^n e^{-x} = 0$, which follows from $e^{-x} = 1/e^x$, and

$$x^{-n}e^{-1/x} \;=\; \frac{x^n}{e^x} \;=\; \frac{x^n}{\sum_{m\ge 0}\frac{x^m}{m!}} \;\le\; \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} \;\longrightarrow\; 0 \qquad (\text{as } x \to +\infty)$$

(This is perhaps a little better than appeals to L'Hospital's Rule.) Thus, f is smooth at 0, with all derivatives 0 there. ///

Next, we make a *smooth bump function* by

$$b(x) \;=\; \begin{cases} 0 & (\text{for } x \le -1) \\ e^{\frac{1}{1-x^2}} & (\text{for } -1 < x < 1) \\ 0 & (\text{for } x \ge 1) \end{cases}$$

A similar argument to the previous shows that this is smooth. Renormalize it to have integral 1 by

$$\beta(x) \;=\; \frac{b(x)}{\int_{-1}^{1} b(t)\;dt}$$

Then $\int_{-1}^{x} \beta(t)\;dt$ is a smooth (monotone) step function that goes from 0 at -1 to 1 at 1. The minor modification $s(x) = 2\int_{-1}^{x}\beta(2t-1)\;dt$ gives a smooth (monotone) step function going from 0 at 0 to 1 at 1. ///

Then $s(x+n+1)$ is a smooth, monoton step function going up from 0 to 1 in $[-n-1, -n]$, and $s(n+1-x)$ for $n \in \mathbb{Z}$ is a smooth, monotone step function going *down* from 1 to 0 in $[n, n+1]$. Thus, the product $f_n(x) = s(x+n+1)\cdot s(n+1-x)$ is the desired smooth cut-off function. ///

[09.2] For a family of smooth cut-off functions $\{f_n\}$ as in the previous example, and for a smooth function g, let $g_n(x) = f_n \cdot g$. Observe that every g_n is a test function. Show that $g_n \to g$ in the $\mathcal{D}^*$-topology.

Discussion: We must show that, for every $\varphi \in \mathcal{D}$,

$$\lim_n \int_{\mathbb{R}} \varphi\cdot(g - g_n) \;=\; 0$$

The point of the smooth cut-off functions f_n is that

$$g(x) - g_n(x) \;=\; g(x)\cdot(1 - f_n(x)) \;=\; g(x)\cdot 0 \;=\; 0 \qquad\qquad (\text{for } x \in [-n,n])$$

Let n_o be sufficiently large so that $\mathrm{spt}\varphi \subset [-n_o, n_o]$. Then for $n \ge n_o$

$$ph(x)\cdot(g(x) - g_n(x)) \;=\; \begin{cases} 0\cdot(g(x) - g_n(x)) & = \;\; 0 \quad (\text{for } |x| \ge) \\ \\ \varphi(x)\cdot 0 & = \;\; 0 \quad (\text{for } |x| \le n) \end{cases}$$

Thus, for each fixed $\varphi \in \mathcal{D}$, the limit 0 is *reached* at the corresponding n_o. ///

[09.3] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$. (Since $\mathscr{S}$ is strictly larger than $\mathcal{D}$, this implies that $\sin(nx) \to 0$ in the $\mathcal{D}^*$-topology.)

Discussion: We must show that, for each $\varphi \in \mathscr{S}$,

$$\lim_n \int_{\mathbb{R}} \sin(nx)\,\varphi(x)\;dx \;\;=\;\; 0$$

On one hand, since Schwartz functions are L^1, we could invoke Riemann-Lebesgue, since (up to normalizations) the indicated integral is $(\widehat{\varphi}(n) - \widehat{\varphi}(-n))/2i$.

On another hand, we also know that $\widehat{\varphi}$ is again a Schwartz function, so $(\widehat{\varphi}(n) - \widehat{\varphi}(-n))/2i \to 0$. (Further, if we know that $\mathscr{S}$ is dense in L^1, then this gives a slightly different proof of Riemann-Lebesgue.) ///

[09.4] Show that $e^{-\varepsilon\pi x^2} \to 1$ as $\varepsilon \to 0^+$ in the $\mathscr{S}^*$ topology. Compute the Fourier transforms of the functions $e^{-\varepsilon\pi x^2}$, and show that they go to δ in the $\mathscr{S}^*$ topology. Obtain the corollary that $\widehat{1} = \delta$ (extended Fourier transform).

Discussion: We must show that, for each $\varphi \in \mathscr{S}$,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} (e^{-\varepsilon\pi x^2} - 1)\,\varphi(x)\;dx \;\;=\;\; 0$$

Since we are accustomed to other uses of ε, let's rewrite this as

$$\lim_{\eta\to 0^+} \int_{\mathbb{R}} (e^{-\eta\pi x^2} - 1)\,\varphi(x)\;dx \;\;=\;\; 0$$

Given $\varepsilon > 0$, for given φ, let N be sufficiently large so that $|\varphi(x)| < \varepsilon/|x|^2$ for $|x| \ge N$. Then

$$\Big|\int_{\mathbb{R}} (e^{-\eta\pi x^2} - 1)\,\varphi(x)\;dx\Big| \;\le\; \int_{|x|\le N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\;dx + \int_{|x|\ge N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\;dx$$

Estimate the second integral:

$$\int_{|x|\ge N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\;dx \;\le\; \int_{|x|\ge N} 2\cdot\frac{\varepsilon}{|x|^2}\;dx \;\le\; \frac{4}{N}\cdot\varepsilon$$

For the first integral, given N and ε, for sufficiently small $\eta > 0$, we have $|e^{-\eta\pi x^2} - 1| < \varepsilon$ for all $|x| \le N$. Thus, $e^{-\eta\pi x^2} \to 1$. ///

Next, the usual trick computes the Fourier transform

$$\widehat{e^{-\eta\pi x^2}}(\xi) \;=\; \frac{1}{\sqrt{\eta}}\cdot e^{-\frac{1}{\eta}\pi\xi^2}$$

The continuity of (extended) Fourier transform $\mathscr{S}^* \longrightarrow \mathscr{S}^*$ assures that

$$(\mathscr{S}^*\text{–}\lim_\eta e^{-\eta\pi x^2})\widehat{\ } \;=\; \mathscr{S}^*\text{–}\lim_\eta \widehat{e^{-\eta\pi x^2}} \;=\; \mathscr{S}^*\text{–}\lim_\eta \frac{1}{\sqrt{\eta}} e^{-\frac{1}{\eta}\pi x^2}$$

For each $\varphi \in \mathscr{S}$,

$$\int_\mathbb{R} \varphi(x) \cdot \frac{1}{\sqrt{\eta}} \cdot e^{-\frac{1}{\eta}\pi x^2}\, dx \;=\; \int_\mathbb{R} \varphi(\sqrt{\eta}\cdot x) \cdot e^{-\pi x^2}\, dx$$

by replacing x by $\sqrt{\eta}\cdot x$. Given $\varepsilon > 0$, let N be large enough so that $|e^{-\pi x^2}| < \varepsilon$ for $|x| \ge N$. Let $\delta > 0$ be small enough so that $|\varphi(x) - \varphi(0)| < \varepsilon$ for $|x| < \delta$. Take $\eta > 0$ sufficiently small so that $N \cdot \sqrt{\eta} < \delta$. Using $\int_\mathbb{R} e^{-\pi x^2}\, dx = 1$,

$$\left|\varphi(0) - \int_\mathbb{R} \varphi(\sqrt{\eta}\cdot x)\cdot e^{-\pi x^2}\, dx\right| \;=\; \left|\int_\mathbb{R} (\varphi(0) - \varphi(\sqrt{\eta}\cdot x))\cdot e^{-\pi x^2}\, dx\right|$$

$$\le \int_{|x|\le N} |\varphi(0) - \varphi(\sqrt{\eta}\cdot x)|\cdot e^{-\pi x^2}\, dx + \int_{|x|\ge N} |\varphi(0) - \varphi(\sqrt{\eta}\cdot x)|\cdot e^{-\pi x^2}\, dx$$

$$< \int_{|x|\le N} \varepsilon \cdot e^{-\pi x^2}\, dx + \int_{|x|\ge N} 2\sup|\varphi|\cdot e^{-\pi x^2}\, dx \;<\; \varepsilon + 2\sup|\varphi|\cdot\varepsilon$$

That is, $\frac{1}{\sqrt{\eta}}\cdot e^{-\frac{1}{\eta}\pi x^2} \longrightarrow 1$ in the $\mathscr{S}^*$ topology. ///

[09.5] Let $-\infty < a < b < c < +\infty$, and

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x < a) \\ A & (\text{for } a < x < b) \\ B & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Show that (extended) $\frac{d}{dx} f = A\delta_a + (B-A)\delta_b - B\delta_c$.

Discussion: This example asks for *proof* of the plausible intuitive idea that a piecewise constant function has derivative 0 along the intervals where it is constant, and multiples of Dirac deltas where jumps occur. There are at least two approaches to the proof, depending whether one characterizes distributions as elements of a dual space, or as $\mathcal{D}^*$-limits of test functions. Granting the theorem that these two characterizations are equivalent, the operational question is which allows an easier approach to the present question.

Perhaps the characterization by duality is more convenient here. Thus, $f' \in \mathcal{D}^*$ is a linear functional on $\mathcal{D}$ characterized by the extension of integration by parts:

$$(\text{as functional})\ f'(\varphi) \;=\; -f(\varphi') \;=\; -\int_\mathbb{R} f(x)\,\varphi'(x)\, dx \qquad (\text{for all } \varphi \in \mathcal{D})$$

Yes, the notation is slightly inconsistent, since on the left f' is a functional on $\mathcal{D}$, in the middle f is a functional on $\mathcal{D}$, while in the integral on the right f is a pointwise-valued function. From the definition of the pointwise-valued function f, integrating by parts or invoking the fundamental theorem of calculus, this is

$$-A\cdot\int_a^b \varphi'(x)\, dx - B\cdot\int_b^c \varphi'(x)\, dx \;=\; -A\cdot(\varphi(b)-\varphi(a)) - B(\varphi(c)-\varphi(b)) \;=\; (A\cdot\delta_a + (B-A)\cdot\delta_b - B\cdot\delta_c)(\varphi)$$

as claimed. ///

[09.6] Let $-\infty < a < b < c < +\infty$, and

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x < a) \\ A_1 x + A_2 & (\text{for } a < x < b) \\ B_1 x + B_2 & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Compute (extended) $\frac{d^2}{dx^2} f$. (It is a linear combination of $\delta_a, \delta'_a, \delta_b, \delta'_b, \delta_c, \delta'_c$.)

Discussion: As in the previous example, this example asks for *proof* of an arguably intuitive approach: in the first derivative, *jumps* produce Dirac deltas, while elsewhere the derivative is (locally) a limit of difference quotients. Thus, the derivative is apparently a sum of locally constant functions and Dirac deltas. Then the second derivative should produce a sum of Dirac deltas from the jumps in the first derivative, and *derivatives* of the Dirac deltas that arose from jumps in the original function.

For the proof, again, the *duality* characterization of $\mathcal{D}^*$ seems more convenient than the characterization of $\mathcal{D}^*$ as a completion of $\mathcal{D}$. Thus, for test function φ, by the dual notion of derivative (compatible with integration by parts, of course)

$$\int_{\mathbb{R}} f'' \cdot \varphi \;=\; \int_{\mathbb{R}} f \cdot \varphi'' \;=\; \int_a^b (A_1 x + A_2) \cdot \varphi''(x)\, dx + \int_b^c (B_1 x + B_2) \cdot \varphi''(x)\, dx$$

The constants A_2 and B_2 and the fundamental theorem of calculus give

$$A_2 \cdot (\varphi'(b) - \varphi'(a)) + B_2 \cdot (\varphi'(c) - \varphi'(b)) \;=\; -A_2 \varphi'(a)) + (A_2 - B_2) \cdot \varphi'(b) - B_2 \varphi'(c)$$

$$=\; \Big(- A_2 \delta'_a + (A_2 - B_2) \cdot \delta'_b - B_2 \delta'_c \Big)(\varphi)$$

The linear terms require an integration by parts: for the first,

$$\int_a^b A_1 x \cdot \varphi''(x)\, dx \;=\; \Big[A_1 x \cdot \varphi'(x) \Big]_a^b - \int_a^b A_1 \cdot \varphi'(x)\, dx \;=\; A_1 (b \cdot \varphi'(b) - a \cdot \varphi'(a)) - A_1 (\varphi(b) - \varphi(a))$$

$$=\; \Big(A_1 (b \delta'_b - a \delta'_a) - A_1 (\delta_b - \delta_a) \Big)(\varphi)$$

by the fundamental theorem of calculus. Similarly,

$$\int_b^c B_1 x \cdot \varphi''(x)\, dx \;=\; \Big[B_1 x \cdot \varphi'(x) \Big]_b^c - \int_b^c B_1 \cdot \varphi'(x)\, dx \;=\; B_1 (c \cdot \varphi'(c) - b \cdot \varphi'(b)) - B_1 (\varphi(c) - \varphi(b))$$

$$=\; \Big(B_1 (c \delta'_c - b \delta'_b) - B_1 (\delta_c - \delta_b) \Big)(\varphi)$$

Thus, altogether,

$$\int_{\mathbb{R}} f'' \cdot \varphi \;=\; \Big(- A_2 \delta'_a + (A_2 - B_2) \cdot \delta'_b - B_2 \delta'_c + A_1 (b \delta'_b - a \delta'_a) - A_1 (\delta_b - \delta_a) B_1 (c \delta'_c - b \delta'_b) - B_1 (\delta_c - \delta_b) \Big)(\varphi)$$

which can be rearranged to suit. ///
