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Examples discussion 10

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[This document is http://www.math.umn.edu/~garrett/m/real/notes_2016-17/real-disc-10.pdf]

[10.1] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, first for $f \in \mathscr{S}(\mathbb{R}^n)$, then for $f \in \mathscr{S}(\mathbb{R}^n)^*$.

Discussion: For $f \in \mathscr{S}(\mathbb{R}^n)$, the literal integral computes the Fourier transform:

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\; g(x)\; dx \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\; f(x + x_o)\; dx$$

Replacing x by $x - x_o$ in the integral gives

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot(x-x_o)}\; f(x)\; dx \;=\; e^{2\pi i\xi\cdot x_o}\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\; f(x)\; dx \;=\; e^{2\pi i\xi\cdot x_o}\cdot \widehat{f}(\xi)$$

The precise corresponding statement for tempered distributions cannot refer to pointwise values. Write ψ_{x_o} for the function $\xi \to e^{2\pi i\xi\cdot x_o}$. Since ψ_{x_o} is bounded, for a tempered distribution u, $\psi_{x_o}\cdot u$ is the tempered distribution described by

$$(\psi_{x_o}\cdot u)(\varphi) \;=\; u(\psi_{x_o}\,\varphi) \qquad\qquad (\text{for } \varphi \in \mathscr{S})$$

This is compatible with multiplication of (integrate-against-) functions $\mathscr{S} \subset \mathscr{S}^*$. Also, let translation $u \to T_{x_o}u$ be defined by $(T_{x_o}u)(\varphi) = u(T_{-x_o}\varphi)$, again compatibly with integration against Schwartz functions. In these terms, the above argument shows that

$$(T_{x_o}f)\widehat{\ } = \;\psi_{x_o}\cdot\widehat{f} \qquad\qquad (\text{for } f \in \mathscr{S})$$

This formulation avoids reference to pointwise values, and thus could make sense for tempered distributions.

One argument is *extension by continuity*: Fourier transform is a continuous map $\mathscr{S}^* \to \mathscr{S}^*$, as is translation $u \to T_{x_o}u$, so the identity extends by continuity to all tempered distributions. ///

Another argument is by *duality*: first,

$$(T_{x_o}u)\widehat{\ }(\varphi) \;=\; (T_{x_o}u)(\widehat{\varphi}) \;=\; u(T_{-x_o}\,\widehat{\varphi}) \;=\; u\Big((\psi_{x_o}\cdot\varphi)\widehat{\ }\Big)$$

by applying the identity to $\varphi, \widehat{\varphi} \in \mathscr{S}$. Going back, this is

$$\widehat{u}(\psi_{x_o}\cdot\varphi) \;=\; (\psi_{x_o}\cdot\widehat{u})(\varphi) \qquad\qquad (\text{for all } \varphi \in \mathscr{S})$$

Altogether, $(T_{x_o}u)\widehat{\ } = \;\psi_{x_o}\cdot\widehat{u}$.

[10.2] Compute $\widehat{\cos x}$.

Discussion: Start from $\widehat{\delta} = 1$. Using the previous example's identity,

$$(T_{x_o}\delta)\widehat{\ } = \;\psi_{x_o}\cdot 1 \;=\; \psi_{x_o}$$

By Fourier inversion, $\widehat{\psi_{x_o}} = T_{-x_o}\delta$. Thus,

$$\widehat{\cos x} \;=\; \tfrac{1}{2}(\psi_{1/2\pi} + \psi_{-1/2\pi})\widehat{\ } = \;\tfrac{1}{2}(T_{-1/2\pi}\delta + T_{1/2\pi}\delta)$$

Written in terms of mock-pointwise-values, this is $\widehat{\cos}(\xi) = \dfrac{\delta(\xi - \frac{1}{2\pi}) + \delta(\xi + \frac{1}{2\pi})}{2}$. ///

[10.3] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\operatorname{spt} \varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x \cdot u = 0$ implies that u is a multiple of δ itself.

Discussion: For $\varphi \in \mathcal{D}$ whose support does *not* include 0, the function $1/x$ is defined and smooth on $\operatorname{spt} \varphi$. Thus, $x \to \varphi(x)/x$ is in $\mathcal{D}$. For such φ,

$$u(\varphi) \;=\; u(x \cdot \frac{\varphi}{x}) \;=\; 0$$

Thus, $\operatorname{spt} u = \{0\}$, so is a finite linear combination $u = \sum_{i=0}^n c_i\, \delta^{(i)}$ with scalars c_i. To see that in fact only δ itself can appear, we use the idea that $1, x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots, \frac{x^n}{n!}$ are essentially a *dual basis* to $\delta, \delta', \delta'', \ldots, \delta^{(n)}$. One way to make this completely precise is to use a smooth cut-off function $\eta \in \mathcal{D}$ around 0, namely, identically 1 on a neighborhood of 0. Then $\eta \cdot x^i \in \mathcal{D}$, and

$$\delta^{(i)}(\eta \cdot \frac{x^j}{j!}) \;=\; \begin{cases} 1 & (\text{for } i = j) \\ \\ 0 & (\text{for } i \neq j) \end{cases}$$

In particular, this shows that the derivatives of δ are *linearly independent*. For $0 \le j \in \mathbb{Z}$,

$$0 \;=\; (x \cdot u)(x^j) \;=\; (x \cdot \sum_i c_i\, \delta^{(i)})(x^j) \;=\; \sum_i c_i\, \delta^{(i)}(x \cdot x^j) \;=\; \sum_i c_i\, \delta^{(i)}(x^{j+1}) \;=\; (j+1)! \cdot c_{j+1}$$

Thus, $c_j = 0$ for $j \ge 1$, and u is a multiple of δ itself. ///

[10.4] Show that the principal value functional $u(\varphi) = P.V. \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$ satisfies $x \cdot u = 1$.

Discussion: For $\varphi \in \mathcal{D}$,

$$u(\varphi) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \ge \varepsilon} \frac{x \cdot \varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \ge \varepsilon} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} 1 \cdot \varphi(x)\, dx \;=\; 1(\varphi)$$

since φ is continuous at 0. Thus, $x \cdot u = 1$. ///

[10.5] Compute the Fourier transform of the sign function

$$\operatorname{sgn}(x) \;=\; \begin{cases} 1 & (\text{for } x > 0) \\ \\ -1 & (\text{for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx}\operatorname{sgn} = 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi i x$, this implies that $(2\pi i)x \cdot \widehat{\operatorname{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x \cdot \widehat{\operatorname{sgn}} = 1$.

Discussion: From the hint, $x \cdot (\pi i\, \widehat{\operatorname{sgn}}) = 1$. Also, the principal-value functional u from the previous example satisfies $x \cdot u = 1$. Thus,

$$x \cdot (u - \pi i\, \widehat{\operatorname{sgn}}) \;=\; 0$$

By another earlier example, this implies that $u - \pi i\, \widehat{\operatorname{sgn}}$ is a multiple of δ. In fact, the multiple is 0, because δ is *even*, while u, sgn, and thus $\widehat{\operatorname{sgn}}$, are all *odd*.[1] That is, $\widehat{\operatorname{sgn}} \;=\; \frac{1}{\pi i} u$. ///

[1] This notion of parity can be defined for distributions from the obvious notion for functions $(\theta \cdot f)(x) = f(-x)$, and then $(\theta \cdot v)(f) = v(\theta \cdot f)$ for distributions v.

[0.1] Remark: In particular, it is *not quite* that $\widehat{\mathrm{sgn}}(\xi) = 1/\pi i\xi$. Indeed, $1/\xi$ is *not* locally integrable, so does not directly describe a distribution. This example shows that, yes, $\xi \cdot \widehat{\mathrm{sgn}} = 1/\pi i$, but apparently we cannot just *divide* (pointwise values). Indeed, we have proven that the principal-value integral is the Fourier transform (up to constants), and it is not quite just an integral.

[10.6] Compute the Fourier transform of $|x|$.

Discussion: From $\frac{d}{dx}|x| = \mathrm{sgn}\, x$, taking Fourier transforms,

$$\widehat{\mathrm{sgn}} \;=\; (\frac{d}{dx}|x|)\widehat{\;} \;=\; 2\pi i \cdot \xi \cdot \widehat{|x|}$$

Recall that in the previous example it was just barely *not* ok to divide by ξ, and the principal-value functional was not quite a literal integral against $1/x$. Similarly, but even more so, here we *cannot* just divide through by ξ to obtain $\widehat{|x|}$ from the principal-value functional.

Similarly, from $(\frac{d}{dx})^2|x| = 2\delta$, by Fourier transform, $(2\pi i)^2 \cdot \xi^2 \cdot \widehat{|x|} = 2 \cdot 1 = 2$ and $-2\pi^2 \cdot \xi^2 \cdot \widehat{|x|} = 1$, but we can't just divide.

We can try to make a $1/x^2$ version of the earlier principal-value functional, such as

$$u(\varphi) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2}\, dx$$

In fact, we can see that this u is the (distributional) derivative of the previous principal-value functional: integrating by parts,

$$\int_{|x|\geq\varepsilon} \frac{\varphi(x) - \varphi(0)}{x^2}\, dx \;=\; \Big[\frac{\varphi(x) - \varphi(0)}{-x}\Big]_\varepsilon^\infty + \Big[\frac{\varphi(x) - \varphi(0)}{-x}\Big]_{-\infty}^{-\varepsilon} - \int_{|x|\geq\varepsilon} \frac{\varphi'(x)}{-x}\, dx$$

$$= -\frac{\varphi(\varepsilon) - \varphi(0)}{-\varepsilon} + \frac{\varphi(-\varepsilon) - \varphi(0)}{-(-\varepsilon)} + \int_{|x|\geq\varepsilon} \frac{\varphi'(x)}{x}\, dx \;=\; \frac{\varphi(\varepsilon) - \varphi(0)}{\varepsilon} - \frac{\varphi(-\varepsilon) - \varphi(0)}{-\varepsilon} + \int_{|x|\geq\varepsilon} \frac{\varphi'(x)}{x}\, dx$$

In the limit, the first two terms give $\varphi'(0) - \varphi'(0) = 0$. Thus, this principal-value functional u is the distributional derivative of the earlier one.

As in the earlier example, we claim that $x^2 \cdot u = 1$: for $\varphi \in \mathcal{D}$,

$$(x^2 \cdot u)(\varphi) \;=\; u(x^2 \cdot \varphi) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{x^2 \cdot \varphi(x) - 0^2 \cdot \varphi(0)}{x^2}\, dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} \varphi(x)\, dx \;=\; 1(\varphi)$$

Thus, both $x^2 \cdot (-2\pi^2 \cdot \widehat{|x|}) = 1$ and $x^2 \cdot u = 1$. Thus, $x^2 \cdot (u - 2\pi^2\, \widehat{|x|}) = 0$. As in an earlier example, this implies that $u - 2\pi^2\, \widehat{|x|}) = a \cdot \delta + b \cdot \delta'$ for some scalars a, b. Since $u, |x|$ and, hence, $\widehat{|x|}$ are *even*, in fact that difference must be a multiple of δ, since δ' is *odd*.

To determine the constant, it suffices to apply both functionals to a convenient $\varphi \in \mathscr{S}$, such as $\varphi(x) = e^{-\pi x^2}$, which is its own Fourier transform. On one hand,

$$u(\varphi) \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{(e^{-\pi x^2})'}{x}\, dx \;=\; \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} \frac{-2\pi x\, e^{-\pi x^2}}{x}\, dx$$

$$= \lim_{\varepsilon\to 0^+} \int_{|x|\geq\varepsilon} -2\pi\, e^{-\pi x^2}\, dx \;=\; \int_{\mathbb{R}} -2\pi\, e^{-\pi x^2}\, dx \;=\; -2\pi$$

On the other hand,

$$\widehat{|x|}(e^{-\pi x^2}) \;=\; |x|(\widehat{e^{-\pi x^2}}) \;=\; |x|(e^{-\pi x^2}) \;=\; \int_{\mathbb{R}} |x| \cdot e^{-\pi x^2}\,dx \;=\; 2\int_0^\infty x e^{-\pi x^2}\,dx \;=\; \int_0^\infty e^{-\pi x}\,dx \;=\; \frac{1}{\pi}$$

by replacing x by $\sqrt{x}$. Thus,

$$a \;=\; a\,\delta(e^{-\pi x^2}) \;=\; (u - 2\pi^2\,\widehat{|x|})(e^{-\pi x^2}) \;=\; (-2\pi) - 2\pi^2 \cdot (\frac{1}{\pi}) \;=\; -2\pi - 2\pi \;=\; -4\pi$$

That is,

$$\widehat{|x|} \;=\; \frac{u}{2\pi^2} + 4\delta$$

... that is, if no constants got lost. ///

[0.2] **Remark:** Again, the principal-value functional u cannot be a literal integral.

[10.7] Determine the Schwartz kernel $K(,)$ for the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$, and show that it is in $H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n})$ for every $\varepsilon > 0$.

Discussion: Let T be the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ viewed as a map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)^*$ via the natural imbedding $\mathcal{D} \subset \mathcal{D}^*$. Write ψ_ξ for the function $\psi_\xi(x) = e^{2\pi i \xi \cdot x}$ for $\xi \in \mathbb{Z}^n$ and $x \in \mathbb{R}/\mathbb{Z}$. Anticipating that K is at worst in $H^{-\infty}(\mathbb{T}^{2n})$, we can write a Fourier expansion $K = \sum_{\xi,\eta\in\mathbb{Z}^n} c_{\xi,\eta}\,\psi_\xi \otimes \psi_\eta$ with coefficients to be determined. [2] Of course there is no reason to think that this converges *pointwise*, and this doesn't matter. The Schwartz kernel for $T : \mathcal{D} \to \mathcal{D}^*$ is characterized by

$$K(\varphi \otimes Tf) \;=\; (Tf)(\varphi) \qquad\qquad (\text{for all } \varphi \in \mathcal{D})$$

Applying this to $\varphi = \psi_\alpha$ and $f = \psi_\beta$,

$$c_{\alpha,\beta} \;=\; K(\psi_\alpha, \psi_\beta) \;=\; (T\psi_\beta)(\psi_\alpha) \;=\; \int_{\mathbb{T}^n} \psi_\beta \cdot \psi_\alpha \;=\; \begin{cases} 0 & (\text{for } \beta \neq -\alpha \in \mathbb{Z}^n) \\ \\ 1 & (\text{for } \beta = \alpha \in \mathbb{Z}^n) \end{cases}$$

The latter *necessary* condition already completely determines K: apparently $K = \sum_\alpha \psi_\alpha \otimes \psi_{-\alpha}$. However, we should give a reason why this expression really does give the identity map on $\mathcal{D}(\mathbb{T}^n)$. Certainly

$$\Big| \sum_{\alpha\in\mathbb{Z}^n} \psi_\alpha \otimes \psi_{-\alpha} \Big|^2_{H^s} \;=\; \sum_{\alpha\in\mathbb{Z}^n} |1|^2 \cdot (1 + |\alpha|^2)^s$$

is finite if and only if $s < -\frac{n}{2}$. Thus, for every $\varepsilon > 0$, $K \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n}) \subset H^{-\infty}(\mathbb{T}^{2n}) = H^\infty(\mathbb{T}^{2n})^*$. That is, that Fourier expansion converges in a Sobolev space and does give a distribution on $\mathbb{T}^{2n}$.

Since finite linear combinations of ψ_α are *dense* in $\mathcal{D}(\mathbb{T}^n)$, and since K is continuous on $H^\infty(\mathbb{T}^n) \otimes H^\infty(\mathbb{T}^n) \subset H^\infty(\mathbb{T}^{2n})$, the earlier computation of $K(\psi_\alpha \otimes \psi_\beta)$ extends by continuity to certify that $K(f \otimes g) = \int f \cdot g$ for $f, g \in \mathcal{D}(\mathbb{T}^n)$. ///

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[2] The tensor notation here is just a way to refer to the function $x, y \to \psi_\xi(x) \cdot \psi_\eta(y)$ without using arguments.