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Examples discussion 11

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[11.1] For $T : V \to V$ a continuous (=bounded) linear map of a Banach space V to itself, show that the operator norm is an upper bound for absolute values of all eigenvalues λ: $|\lambda|_{\mathbb{C}} \le |T|_{\rm op}$. Further, show that $|T|_{\rm op}$ is an upper bound for *all* of the spectrum, that is, $T - \lambda$ is invertible for $|\lambda|_{\mathbb{C}} > |T|_{\rm op}$.

Discussion: First, for $Tv = \lambda \cdot v$ for $0 \ne v \in V$, without loss of generality take $|v| = 1$, and then

$$|T|_{\rm op} \;=\; \sup_{|w|\le 1} |Tw| \;\ge\; |Tv| \;=\; |\lambda \cdot v| \;=\; |\lambda|_{\mathbb{C}} \cdot |v|_V \;=\; |\lambda|$$

Second, for $|\lambda|_{\mathbb{C}} > |T|_{\rm op}$, we have $|T/\lambda|_{\rm op} < 1$, so

$$S \;=\; 1 + T/\lambda + (T/\lambda)^2 + \ldots \;=\; \lim_N (1 + T/\lambda + (T/\lambda)^2 + \ldots + (T/\lambda)^N)$$

is convergent in the operator norm on the Banach space of continuous/bounded linear operators on V, since the tails go to 0. Since $1 - T/\lambda$ is continuous, as expected

$$(1-T/\lambda)\cdot S \;=\; S\cdot(1-T/\lambda) \;=\; \lim_N (1-T/\lambda)\cdot(1+T/\lambda+(T/\lambda)^2+\ldots+(T/\lambda)^N) \;=\; \lim_N 1-(T/\lambda)^{N+1} \;=\; 1$$

since $(T/\lambda)^N \to 0$: if there were any doubt,

$$|(T/\lambda)^N|_{\rm op} \;\le\; |T/\lambda|^N \;\longrightarrow\; 0$$

since $|T/\lambda|_{\rm op} < 1$. Thus, the inverse $S = (1 - T/\lambda)^{-1}$ exists (as a continuous linear operator), and λ is *not* in any part of the spectrum. ///

[11.2] *(Weyl's criterion: approximate eigenvectors and continuous spectrum)* Let $T : V \to V$ be a continuous linear operator on a Hilbert space V. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T - \lambda)v_n \to 0$ as $n \to +\infty$ is an *approximate eigenvector* for λ. Show that for λ *not* an eigenvalue for T, λ has an approximate eigenvector if and only if λ is in the spectrum of T.

Discussion: In fact, this criterion is *not* reliable for detecting some types of *residual* spectrum. [1] Specifically, for λ not an eigenvector, we need to assume that $(T - \lambda)V$ is *not closed* for existence of approximate eigenvectors to imply that λ is in the spectrum. This is implied by T being *normal*, for example *self-adjoint*. We give an example at the end of the discussion where this criterion fails.

First, certainly, if λ is an eigenvector, with non-zero eigenvalue v, the constant sequence $v, v, v, \ldots$ fits the requirement.

For general spectrum, let $S = T - \lambda$. For $v_1, v_2, \ldots$ with $|v_n| = 1$ and $Sv_n \to 0$, any alleged (continuous[2]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 \;=\; S^{-1}(\lim_n Sv_n) \;=\; \lim_n S^{-1}Sv_n \;=\; \lim_n v_n$$

[1] Recall that *residual* spectrum of T is λ such that $T - \lambda$ is *injective*, but does *not* have dense image.

[2] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S, necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta > 0$ such that $|Sv| \ge \delta \cdot |v|$ for all v. This exactly asserts the boundedness of S^{-1}, so S^{-1} is *continuous*.

contradiction. Thus, existence of an approximate eigenvector for $T - \lambda$ implies that $T - \lambda$ is not invertible.

Conversely, for $S = T - \lambda$ not invertible, but λ not an eigenvector, then S is *injective* but not *surjective*. As noted above, we must assume that the image of S is *not closed*. In that case, S is injective, not surjective, and by non-closedness of the image there is v_o (with $|v_o| = 1$) not in the image of S, and $v_1, v_2, \ldots$ such that $Sv_1, Sv_2, \ldots \to v_o$.

[11.1] Theorem: For λ *not* an eigenvalue for T, and for $(T - \lambda)V$ *not closed*, λ is in the spectrum of T if and only if λ has an approximate eigenvector.

[11.2] Remark: This criterion is *not* uniformly reliable for detecting *residual* spectrum, which is why we must impose a further condition. [3] For example, we have seen that, for $T : V \to V$ a *norma* linear operator, for λ in the spectrum but not an eigenvalue, $(T - \lambda)V$ is *dense* in V but is not all of V. Thus, the hypothesis of the theorem is met for normal T. We give an example of failure to detect residual spectrum after the proof.

Proof: Certainly if λ is an eigenvector, with non-zero eigenvalue v, the constant sequence $v, v, v, \ldots$ fits the requirement.

For general spectrum, let $S = T - \lambda$. For $v_1, v_2, \ldots$ with $|v_n| = 1$ and $Sv_n \to 0$, any alleged (continuous [4]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 \;=\; S^{-1}(\lim_n Sv_n) \;=\; \lim_n S^{-1}Sv_n \;=\; \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T - \lambda$ implies that $T - \lambda$ is not invertible.

Conversely, for $S = T - \lambda$ not invertible, but λ not an eigenvector, then S is *injective* but not *surjective*. We further assume that the image of S is *not closed*. [5] In that case, S is injective, not surjective, and by non-closedness of the image there is v_o (with $|v_o| = 1$) not in the image of S, and $v_1, v_2, \ldots$ such that $Sv_1, Sv_2, \ldots \to v_o$. If $\{v_n\}$ were a Cauchy sequence, then it would have a limit, and by continuity of S

$$v_o \;=\; \lim_n Sv_n \;=\; S(\lim_n v_n)$$

and v_o would be in the image of S, contradicting our assumption. Thus, $\{v_n\}$ is *not* Cauchy. In particular, we can replace $\{v_n\}$ by a subsequence such that there is $\delta > 0$ such that $|v_m - v_n| \geq \delta$ for all $m \neq n$. Then $w_n = v_n - v_{n+1}$ forms an approximate 0-eigenvector, since their lengths are bounded away from 0, and

$$Sw_n \;=\; S(v_n - v_{n+1}) \;=\; Sv_n - Sv_{n+1} \;\longrightarrow\; v_o - v_o \;=\; 0$$

as desired. ///

As noted, the case that λ is not an eigenvector, $T - \lambda$ is not surjective, and/but the image of $S = T - \lambda$ is *closed*, can only occur for non-normal T. For example, $T : \ell^2 \to \ell^2$ by

$$T(c_1, c_2, \ldots) \;=\; (c_1, 0, c_2, 0, c_3, 0, \ldots)$$

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[3] Recall that *residual* spectrum of T is λ such that $T - \lambda$ is *injective*, but does *not* have dense image.

[4] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S, necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta > 0$ such that $|Sv| \geq \delta \cdot |v|$ for all v. This exactly asserts the boundedness of S^{-1}, so S^{-1} is *continuous*.

[5] The image is not closed, for example, when T (hence S) has no residual spectrum, which is the case when T (hence S) is *normal*, or *self-adjoint*.

is injective, not surjective, and has closed image. It is not invertible, but there is no approximate eigenvector for 0, so the criterion fails in this (non-normal) example. ///

[11.3] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x) \cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectum and continuous spectrum, that is, no left-over *residual* spectrum.)

Discussion: This operator is self-adjoint, since $\sin x$ is real-valued:

$$\langle Tf, g\rangle \;=\; \int_{\mathbb{R}} f(x) \cdot \sin x \cdot \overline{g(x)}\, dx \;=\; \int_{\mathbb{R}} f(x) \cdot \overline{g(x) \cdot \sin x}\, dx \;=\; \langle f, Tg\rangle$$

For a function f and fixed $\lambda \in \mathbb{C}$ such that $f(x) \cdot \sin x = \lambda \cdot f(x)$ for almost all x, for x such that $f(x) \neq 0$, necessarily $\sin x = \lambda$. Since $\sin x$ assumes any particular value at most countably many times, $f = 0$ almost everywhere. Thus, there are no eigenvalues.

Since T is self-adjoint, it is normal, so there is no residual spectrum. Thus, Weyl's criterion via approximate eigenvectors suffices to determine the remainder of the spectrum, which will be *continuous*. Given a value $\lambda \in [-1, 1]$, let $x_o \in \mathbb{R}$ be such that $\sin x_o = \lambda$. We claim that an approximate eigenvector for λ can be formed by functions concentrated ever-more-closely at x_o, such as

$$v_n(x) \;=\; \begin{cases} \sqrt{n} & (\text{for } |x - x_o| \le \frac{1}{2n}) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

By design, $|v_n| = 1$. Since $\sin x$ is continuous, given $\varepsilon > 0$ there is $\delta > 0$ such that $|\sin x - \sin x_o| < \varepsilon$ for $|x - x_o| < \delta$. For n large enough so that $1/2n < \delta$,

$$|Tv_n - \lambda v_n|^2_{L^2} \;=\; |v_n \cdot \sin x - \lambda \cdot v_n|^2_{L^2} \;=\; \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot |\sin x - \sin x_o|^2\, dx \;<\; \int_{x_o - \frac{1}{2n}}^{x_o + \frac{1}{2n}} n \cdot \varepsilon^2\, dx \;=\; \varepsilon^2$$

Thus, $Tv_n - \lambda v_n \to 0$, and $\{v_n\}$ is an approximate identity for λ, so every $\lambda \in [-1, 1]$ is in the continuous spectrum. ///

[11.4] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0, 1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0, 1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

Discussion: Since the set $\{|r_1|, |r_2|, \ldots\}$ is bounded by 1, the operator norm of T is at most 1, so it is bounded, hence continuous. Since the r_n are all *real*, the operator is self-adjoint:

$$\langle T(a_1, a_2, \ldots),\, (b_1, b_2, \ldots)\rangle \;=\; \langle (r_1 a_1, r_2 a_2, \ldots),\, (b_1, b_2, \ldots)\rangle \;=\; \sum_n r_n a_n \cdot \overline{b_n}$$

$$=\; \sum_n a_n \cdot \overline{r_n b_n} \;=\; \langle (a_1, a_2, \ldots),\, T(b_1, b_2, \ldots)\rangle$$

When $\lambda \cdot (c_1, c_2, \ldots) = T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$, necessarily $\lambda \cdot c_n = r_n \cdot c_n$ for all n. When $c_n \neq 0$, this implies $\lambda = r_n$. Since the r_n are distinct, there can be (at most) one index n for which $c_n \neq 0$, and then $\lambda = r_n$. Conversely, every r_n is obviously an eigenvalue.

Since we know that the whole spectrum is *closed* in $\mathbb{C}$, it contains at least the closure of the rationals in $[0,1]$, namely, $[0,1]$ itself. Since T is self-adjoint, its spectrum is contained in $\mathbb{R}$. [6] Since the spectrum is bounded by $|T|_{\text{op}} = 1$, it is contained in $[-1,1]$.

To see that $\lambda \in [-1,0)$ is *not* in the spectrum, in that $(T-\lambda)(c_1,c_2,\ldots) = ((r_1-\lambda)c_1,(r_2-\lambda)c_2,\ldots)$, we have $|r_n - \lambda| \ge |\lambda| > 0$, so the inverse $(T-\lambda)^{-1}$ can be written down immediately: $(T-\lambda)^{-1}(c_1,c_2,\ldots) = ((r_1-\lambda)^{-1}c_1,\, (r_2-\lambda)^{-1}c_2,\ldots)$ and there is a uniform upper bound $|(r_n-\lambda)^{-1}| \le |\lambda|^{-1}$. [7] Finally, given irrational $\lambda \in [0,1]$, let $r_{n_1}, r_{n_2},\ldots$ be rationals such that $r_{n_i} \to \lambda$. With standard basis $\{e_n\}$ for ℓ^2, we claim that $\{e_{n_i}\}$ is an approximate eigenvector for λ: given $\varepsilon > 0$, let N be sufficiently large so that $|r_{n_i} - \lambda| < \varepsilon$ for $i \ge N$. For $n_i \ge N$,

$$|(T-\lambda)e_{n_i}| \;=\; |(r_{n_i}-\lambda)e_{n_i}| \;=\; |r_{n_i}-\lambda|_{\mathbb{C}} \cdot |e_{n_i}|_{\ell^2} \;=\; |r_{n_i}-\lambda|_{\mathbb{C}} \;<\; \varepsilon$$

Thus, indeed, $(T-\lambda)e_{n_i} \to 0$, and the e_{n_i} give an approximate identity for λ, so λ is in the spectrum.
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[11.5] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1,c_2,\ldots) = (r_1c_1, r_2c_2,\ldots)$. Show that T is *compact* if and only if $r_n \to 0$.

Discussion: Let $e_1, e_2, \ldots$ be the standard (Hilbert-space) basis for ℓ^2. If the r_n do not go to 0, then there is a *subsequence* $r_{n_1}, r_{n_2},\ldots$ bounded away from 0. Since T is compact, the images $Te_{n_i} = r_{n_i}e_{n_i}$ must have a convergent subsequence. But $|r_{n_i}e_{n_i} - r_{n_j}e_{n_j}|^2 = |r_{n_i}|^2 + |r_{n_j}|^2$ for $i \ne j$, and this is bounded away from 0, so there is no convergent subsequence, contradicting the compactness of T. Thus, in fact, $r_n \to 0$.

For the converse, perhaps the most economical approach is to observe that T is an operator-norm limit of finite-rank operators, hence compact:

$$T_n(c_1,c_2,\ldots,c_n,c_{n+1},\ldots) \;=\; (c_1,c_2,\ldots,c_n,0,0,\ldots)$$

The estimate on the operator norms is

$$|T - T_n|_{\text{op}} \;=\; \sup_{|v|\le 1} |(0,\ldots,0,r_{n+1}v_{n+1},\ldots)| \;=\; \sup_{k\ge n} |r_k| \; To\, 0$$

Less efficiently, we can refer to definitions, and use the *total boundedness* criterion for compact closure. Given $\varepsilon > 0$, let N be large enough so that $|r_n| < \varepsilon$ for $n \ge N$. Write $v = (v_1,v_2,\ldots) \in \ell^2$ as

$$v \;=\; \underbrace{(v_1,\ldots,v_N,0,0,\ldots)}_{v'} + \underbrace{(\underbrace{0,\ldots,0}_{N},v_{N+1},\ldots,v_{N+2},\ldots)}_{v''}$$

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[6] The proof that self-adjoint operators T have spectrum inside $\mathbb{R}$ has more content than just the analogous assertion about eigenvectors. For $Tv = \lambda v$ with $v \ne 0$, of course

$$\lambda\langle v,v\rangle \;=\; \langle \lambda v,v\rangle \;=\; \langle Tv,v\rangle \;=\; \langle v,Tv\rangle \;=\; \langle v,\lambda v\rangle \;=\; \overline{\lambda}\langle v,v\rangle$$

shows that any eigenvalues are real. Since self-adjoint operators have no residual spectrum, to find the rest of the spectrum it suffices to identify *approximate eigenvectors*. Note that for self-adjoint T always $\langle Tv,v\rangle = \overline{\langle v,Tv\rangle} = \overline{\langle Tv,v\rangle}$, so $\langle Tv,v\rangle$ is *real*. Then for $(T-\lambda)v_n \to 0$, certainly $\langle (T-\lambda)v_n, v_n\rangle \to 0$, so the *imaginary parts* go to 0. These are

$$\text{Im}\,\langle (T-\lambda)v_n,v_n\rangle \;=\; \text{Im}\,\langle Tv_n,v_n\rangle + \text{Im}\,(\lambda\cdot\langle v_n,v_n\rangle) \;=\; 0 + \text{Im}\,(\lambda)\cdot\langle v_n,v_n\rangle$$

Since $|v_n|$ are bounded away from 0, there can be an approximate identity only for $\lambda \in \mathbb{R}$. ///

[7] For such a simple operator, a similar device shows that $\lambda \notin \mathbb{R}$ is not in the spectrum.

Let B' be the intersection of the unit ball $B \subset \ell^2$ with the copy of $\mathbb{C}^N \subset \ell^2$ with non-zero components only at the first N places. Let B'' be the intersection of B with the subspace of ℓ^2 with 0 entries at the first N places. Certainly $B' + B'' \supset B$ and $B' \perp B''$.

By design, $|Tv''| \le \varepsilon$ for $v'' \in B''$. Since TB' is a bounded subset of a finite-dimensional space $\mathbb{C}^N$, it has compact closure, so is totally bounded, so can be covered by finitely-many ε-balls $U_1, \ldots, U_k$. Then $TB \subset TB' + TB'' \subset (U_1 + TB'') \cup \ldots \cup (U_k + TB'')$, and every $U_i + TB''$ is contained in a 2ε-ball. Thus, TB is totally bounded, hence, has compact closure. ///

[11.6] Let V be the Volterra operator $Vf(x) = \int_0^x f(t)\,dt$ on $L^2[0,1]$. Show that $|V^n|_{\text{op}} \to 0$ as $n \to +\infty$. Show that the spectrum of V is just $\{0\}$.

Discussion: In fact, $|V^n|_{\text{op}} \to 0$ is a weaker conclusion than was intended, in part because it would *not* help showing that the spectrum of V is just $\{0\}$. Rather we would want to show something like $\lim_n |V^n|_{\text{op}}^{1/n} = 0$.

Nevertheless, one way to estimate the behavior of V^n is to show that that $|V|_{\text{op}} < 1$. For the latter, recall that a linear map $T : L^2(X) \to L^2(X)$ given by $K(,) \in L^2(X \times X)$ by

$$Tf(x) \;=\; \int_X K(x,y)\, f(y)\; dy$$

is a Hilbert-Schmidt operator, with operator norm bounded by the $L^2(X \times X)$ norm of $K(,)$ (this is the *Hilbert-Schmidt norm* of the operator). Thus,

$$|V|_{\text{op}}^2 \;\le\; \int_0^1 \int_0^1 \left| \begin{cases} 1 & (\text{for } y < x) \\ 0 & (\text{for } y > x) \end{cases} \right|^2 dx\,dy \;=\; \int_0^1 \int_0^x 1\; dx\,dy \;=\; \int_0^1 x\; dx \;=\; \tfrac{1}{2} \;<\; 1$$

Thus, certainly, $|V^n|_{\text{op}} \le |V|_{\text{op}}^n \to 0$. ///

We also recall the argument that V has no *eigenvalues*: when $\int_0^x f(t)\,dt = \lambda \cdot f(x)$ for $\lambda \neq 0$, application of Cauchy-Schwarz-Bunyakowsky to the left-hand side shows that f is *continuous*. Then the equation shows that $v \in C^1$ (and induction shows that $f \in C^\infty$). Differentiating the eigenfunction condition, $f = \lambda \cdot f'$. We know how to solve this equation: all solutions are multiples of $x \to e^{x/\lambda}$. However, the eigenfunction relation also shows that $f(0) = 0$ (and f is continuous, so this holds in a strong sense: we are not allowed to change its values on sets of measure 0), which does not hold for any of these functions. Thus, there are no eigenvalues. ///

Now we return to demonstration of the stronger result, that $|V^n|_{\text{op}}^{1/n} \to 0$. To do so, we consider the integral/Schwartz kernel for the iterate V^n:

$$(V^n f)(x) \;=\; \int_0^x \int_0^{x_{n-1}} \cdots \int_0^{x_2} \int_0^{x_1} f(y)\; dy\, dx_1\, dx_2\, \ldots\, dx_{n-2}\, dx_{n-1}$$

Changing the order of integration to isolate the kernel, this is

$$(V^n f)(x) \;=\; \int_0^x f(y) \Big(\int_y^x \int_y^{x_{n-1}} \cdots \int_y^{x_2} 1\, dx_1\, dx_2\, \ldots\, dx_{n-2}\, dx_{n-1} \Big)\; dy$$

By induction, the inner integral is $\dfrac{(x-y)^{n-1}}{(n-1)!}$ for $0 \le y \le x$. That is, the kernel $K_n(x,y)$ for V^n is

$$K_n(x,y) \;=\; \begin{cases} \dfrac{(x-y)^{n-1}}{(n-1)!} & (\text{for } y < x) \\ 0 & (\text{for } y > x) \end{cases}$$

and its L^2 norm squared is

$$\int_0^1 \int_0^x |K_n(x,y)|^2 \, dy \, dx \;=\; \int_0^1 \int_0^x \frac{(x-y)^{2(n-1)}}{((n-1)!)^2} \, dy \, dx$$

$$= \int_0^1 \frac{x^{2n-1)}}{((n-1)!)^2 \, (2n-1)} \, dx \;=\; \frac{1}{((n-1)!)^2 \, (2n-1) \, (2n)} \;<\; \Big(\frac{1}{n!}\Big)^2$$

Thus, its L^2 norm is bounded by $1/n!$. Since[8] $(n!)^{1/n} \to +\infty$, the spectral radius is 0. ///

To show that $V - \lambda$ is invertible for all $\lambda \neq 0$, it is *not* sufficient to know that $|V|_{\text{op}} < 1$ nor that $|V^n|_{\text{op}} \to 0$, but knowing $|V^n|_{\text{op}}^{1/n} \to 0$ does suffice, as follows.

To show $T - \lambda = -\lambda \circ (1 - T/\lambda)$ is invertible for $\lambda \neq 0$, it suffices to show that the series

$$1 + T/\lambda + (T/\lambda)^2 + (T/\lambda)^3 + \ldots$$

for the obvious candidate for $(1 - T/\lambda)^{-1}$ converges in operator norm. That is, we want

$$1 + |T/\lambda|_{\text{op}} + |(T/\lambda)^2|_{\text{op}} + |(T/\lambda)^3|_{\text{op}} + \ldots \;<\; +\infty$$

The left-hand side is

$$1 + |T|_{\text{op}} \cdot |\lambda|^{-1} + |T^2|_{\text{op}} \cdot |\lambda|^{-2} + |T^3|_{\text{op}} \cdot |\lambda|^{-3} + \ldots$$

Applying the *root test*,

$$\limsup_n \Big(|T^n|_{\text{op}} \cdot |\lambda|^{-n}\Big)^{1/n} \;\leq\; \limsup_n (|T^n|_{\text{op}})^{1/n} \cdot |\lambda|^{-1} \;=\; |\lambda|^{-1} \cdot \limsup_n (|T^n|_{\text{op}})^{1/n} \;\leq\; |\lambda|^{-1} \cdot 0 \;<\; 1$$

so the series converges for all $\lambda \neq 0$. That is, $(T - \lambda)^{-1}$ exists for all $\lambda \neq 0$. ///

[11.3] Remark: For self-adjoint (or, more generally, *normal*) operators T, in fact $\lim_n |T^n|_{\text{op}}^{1/n} = |T|_{\text{op}}$. However, as we see for the Volterra operator, in general $\lim_n |T^n|_{\text{op}}^{1/n} \leq |T|_{\text{op}}$. For not-necessarily-normal operators, the *spectral radius* is $\limsup_n |V^n|_{\text{op}}^{1/n}$. The argument given for the Volterra operator shows that in general the spectral radius is an upper bound for the spectrum.

[11.4] Remark: In fact, the spectral radius is a *sharp* bound for the absolute values $|\lambda|$ for λ in the spectrum. *[... iou ...]*

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[8] That $(n!)^{1/n} \to +\infty$ certainly follows from Stirling's asymptotic, but also more elementary considerations. For example, for each fixed $1 \leq k \in \mathbb{Z}$, for $n \geq k$

$$\Big(\frac{1}{n!}\Big)^{1/n} \;\leq\; (k^{n-k+1})^{1/n} \;=\; k^{1-\frac{k-1}{n}} \;\longrightarrow\; k$$

Since this holds for every fixed k, the limit must be $+\infty$.