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Examples discussion 12

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-disc-12.pdf]

[12.1] Let T be a compact operator $T : V \to W$ for Hilbert spaces V, W. For S a continuous/bounded operator on V, show that $T \circ S : V \to W$ is compact. For R a continuous/bounded operator on W, show that $R \circ T : V \to W$ is compact.

Discussion: For $T \circ S$, the image of the unit ball under S is contained in some ball $c \cdot B$, where B is the unit ball, because S is bounded. Since T is linear, $T(c \cdot B) = c \cdot TB$. Since TB is pre-compact, its continuous image under multiplication by c is also pre-compact. Proof: for $c = 0$, we're done. For $c > 0$, given a finite cover of TB by balls $w_i + B_\varepsilon$ where B_ε is the ball of radius $\varepsilon > 0$ centered at 0. The images $c \cdot (w_i + B_\varepsilon) = cw_i + cB_i$ cover $c \cdot TB$, and have radius $c \cdot \varepsilon$. Replacing ε by ε/c gives balls of radius ε covering $c \cdot TB$. ///

For $R \circ T$, similarly as in the previous case, given a finite cover of TB by balls $w_i + B_\varepsilon$ of radius $\varepsilon > 0$, the images $R(w_i + B_\varepsilon) = Rw_i + RB_\varepsilon$ are contained in balls $Rw_i + cB_\varepsilon$, where $c = |R|_{\text{op}}$ will suffice. ///

[12.2] (*Rellich's lemma on the circle*) For $s < t \in \mathbb{R}$, show that the inclusion map $H^t(\mathbb{T}) \to H^s(\mathbb{T})$ is *compact*. (*Hint:* Use the orthogonal bases $\psi_n(x) = e^{2\pi i n x}$, and note that their lengths in $H^s(\mathbb{T})$ vary depending on s. Thus, if we choose isomorphisms of H^s and H^t to $\ell^2(\mathbb{Z})$, the inclusion $H^t \to H^s$ sending $\psi_n \to \psi_n$ will *not* be the identity map on those copies of $\ell^2(\mathbb{Z})$.)

Discussion: The H^s-norm of ψ_n is $(1+n^2)^s$, so $\psi_n/(1+n^2)^s$ is an orthonormal basis. The identity map sends

$$\frac{\psi_n}{(1+n^2)^t} \longrightarrow \frac{\psi_n}{(1+n^2)^t} = \frac{\psi_n}{(1+n^2)^s} \cdot (1+n^2)^{s-t}$$

Thus, viewed as a map of $\ell^2(\mathbb{Z})$ to itself, it multiplies the n^{th} element of an orthonormal basis by $(1+n^2)^{s-t}$. For $t > s$, these go to 0, and we know that this implies that the map is compact. ///

[12.3] Let $K(,)$ be a measurable function on $\mathbb{R}^2$, with a bound B such that $\int_\mathbb{R} |K(x,y)|\,dx \le B$ for every y, and $\int_\mathbb{R} |K(x,y)|\,dy \le B$ for every x. Show that $Tf(x) = \int_\mathbb{R} K(x,y)\,f(y)\,dy$ gives a continuous linear map $L^p \to L^p$ for every $1 < p < \infty$, with $|Tf|_{L^p} \le B \cdot |f|_{L^p}$. (*Hint:* Hölder's inequality.)

Discussion: There is a slight further algebraic trick beyond just Hölder's inequality, manifest in the following: letting q be the *dual exponent* so that $\frac{1}{p} + \frac{1}{q} = 1$,

$$|Tf(x)| = \Big| \int_\mathbb{R} K(x,y)\,f(y)\,dy \Big| \le \int_\mathbb{R} |K(x,y)|^1\,|f(y)|\,dy = \int_\mathbb{R} |K(x,y)|^{\frac{1}{q}} \cdot |K(x,y)|^{\frac{1}{p}} \cdot |f(y)|\,dy$$

$$\le \Big(\int_\mathbb{R} (|K(x,y)|^{\frac{1}{q}})^q\,dy \Big)^{1/q} \cdot \Big(\int_R \Big(|K(x,y)|^{\frac{1}{p}} \cdot |f(y)| \Big)^p\,dy \Big)^{1/p}$$

$$= \Big(\int_\mathbb{R} |K(x,y)|\,dy \Big)^{1/q} \cdot \Big(\int_R |K(x,y)| \cdot |f(y)|^p\,dy \Big)^{1/p} \le B^{1/q} \cdot \Big(\int_R |K(x,y)| \cdot |f(y)|^p\,dy \Big)^{1/p}$$

invoking Hölder. Using this and invoking Fubini-Tonelli,

$$|Tf|^p_{L^p} = \int_\mathbb{R} \Big| \int_\mathbb{R} K(x,y)\,f(y)\,dy \Big|^p\,dx \le \int_\mathbb{R} B^{p/q} \int_\mathbb{R} |K(x,y)| \cdot |f(y)|^p\,dy\,dx$$

$$= B^{p/q} \int_\mathbb{R} |f(y)|^p \Big(\int_\mathbb{R} |K(x,y)|\,dx \Big)\,dy \le B^{p/q} \int_\mathbb{R} |f(y)|^p \cdot B\,dy = B^{\frac{p}{q}+1} \cdot |f|^p_{L^p}$$

Since $(B^{\frac{p}{q}+1})^{1/p} = B^{\frac{1}{p}+\frac{1}{q}} = B^1 = B$, we have the assertion. ///

[12.4] (A simple instance of *Young's inequality*) In the previous example, let $K(x,y) = k(x-y)$ for $k \in L^1(\mathbb{R})$, so that $Tf(x) = (k * f)(x)$. Show that $|Tf|_{L^p} \le |k|_{L^1} \cdot |f|_{L^p}$.

Discussion: A simple case of the previous. ///

[12.5] Solve $-u'' + u = \delta$ on $\mathbb{R}$. (*Hint:* use Fourier transform. Knowing how to evaluate standard/iconic integrals by residues would be convenient, but/and the relevant integral was done in an earlier example-discussion.)

Discussion: Let's assume that we are asking for a solution u that is at worst a tempered distribution. Thus, we can take Fourier transform, obtaining

$$(4\pi^2\xi^2 + 1)\widehat{u} \;=\; \widehat{\delta} \;=\; 1$$

Obviously we want to *divide* by $4\pi^2\xi^2 + 1$. Unlike some other examples, where division was not quite legitimate, here, we can achieve the effect by *multiplication* by the smooth, bounded function $1/(4\pi^2\xi^2+1)$, since $4\pi^2\xi^2 + 1$ does not vanish on $\mathbb{R}$. Thus,

$$\widehat{u} \;=\; \frac{1}{4\pi^2\xi^2 + 1}$$

Since the right-hand side is luckily in $L^1(\mathbb{R})$, we can compute its image under Fourier inversion by the literal integral, its inverse Fourier transform will be a continuous function (by Riemann-Lebesgue), so has meaningful pointwise values:

$$u(x) \;=\; \int_{\mathbb{R}} \frac{e^{2\pi i\xi x}}{4\pi^2\xi^2 + 1}\, d\xi$$

The integral can be evaluated by *residues*: depending on the sign of x, we use an auxiliary arc in the upper (for $x > 0$) or lower (for $x < 0$) half-plane, so that $\xi \to e^{2\pi i\xi x}$ is *bounded* in the corresponding half-plane. Thus, we pick up either $2\pi i$ times the residue at $\xi = 1/2\pi i$, or the negative (because the orientation is negative) of the residue at $\xi = -1/2\pi i$. That is, respectively,

$$2\pi i \cdot \frac{e^{2\pi i\cdot(1/2\pi i)\cdot x}}{4\pi^2 \cdot (\frac{1}{2\pi i} - \frac{-1}{2\pi i})} \;=\; -e^{-x} \;=\; -e^{-|x|} \qquad\qquad (\text{for } x \ge 0)$$

and

$$-2\pi i \cdot \frac{e^{2\pi i\cdot(-1/2\pi i)\cdot x}}{4\pi^2 \cdot (\frac{-1}{2\pi i} - \frac{1}{2\pi i})} \;=\; -e^{x} \;=\; -e^{-|x|} \qquad\qquad (\text{for } x \le 0)$$

[12.6] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution. (And reflect on the Fredholm alternative?)

Discussion: In Fourier series converging in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for all $\varepsilon > 0$, $\delta_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}} 1\cdot\psi_n$, where $\psi_n(x) = e^{2\pi inx}$. A function u in the relatively large-yet-tractable space $H^{-\infty}(\mathbb{T})$ has a Fourier expansion $u = \sum_n \widehat{u}(n)\cdot\psi_n$. Application of the (extended-sense) second derivative operator can be done termwise (by design), and annihilates the $n = 0$ term. That is, no u'' can have 0^{th} Fourier coefficient 1, as does $\delta_{\mathbb{Z}}$, so that equation is not solvable. ///

In contrast, $\delta_{\mathbb{Z}} - 1$ has exactly lost that difficult Fourier component, and, in terms of Fourier series, $u'' = \delta_{\mathbb{Z}} - 1$ is

$$\sum_{n\in\mathbb{Z}} (2\pi in)^2) \cdot \widehat{u}(n) \cdot \psi_n \;=\; \sum_{n\ne 0} 1\cdot\psi_n$$

has the solution *by division*

$$u \;=\; \sum_{n\ne 0} \frac{1}{(2\pi in)^2}\psi_n$$

[12.7] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1. (And reflect on the Fredholm alternative?)

Discussion: The orthogonality to 1 means that the 0^{th} Fourier coefficient of f is 0. Thus, on the Fourier series side, for any $u \in H^{-\infty}(\mathbb{T})$, $u'' = f$ is

$$\sum_{n\in\mathbb{Z}} (2\pi in)^2 \cdot \widehat{u}(n) \cdot \psi_n \;=\; \sum_{n\neq 0} \widehat{f}(n) \cdot \psi_n$$

gives

$$u \;=\; \sum_{n\neq 0} \frac{\widehat{f}(n)}{(2\pi in)^2} \cdot \psi_n$$

and there is no other solution in $H^{-\infty}(\mathbb{T})$. ///
