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Examples discussion 13

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-disc-13.pdf]

[13.1] On $\mathbb{R}^2$, compute the Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2+y^2)}$ for $n = 0, 1, 2, \ldots$. (*Hint:* Re-express things, including Fourier transform, in terms of $z = x + iy$ and $\overline{z} = x - iy$, $w = u + iv$, and $\overline{w} = u - iv$.)

Discussion: Using z and w, the functions are $z^n e^{-\pi z\overline{z}}$ and $\overline{z}^n e^{-\pi z\overline{z}}$, and Fourier transform is

$$\int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, z^n e^{-\pi z\overline{z}}\, dx\, dy \;=\; \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, \frac{1}{(-\pi)^n}\Big(\frac{\partial}{\partial \overline{z}}\Big)^n e^{-\pi z\overline{z}}\, dx\, dy$$

Imagining that we can integrate by parts, this is

$$(-1)^n \frac{1}{(-\pi)^n}\int_{\mathbb{R}^2} \Big(\frac{\partial}{\partial \overline{z}}\Big)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; \frac{1}{\pi^n}\int_{\mathbb{R}^2} (-\pi i w)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy$$

$$=\; (-i)^n\, w^n \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; i^{-n}\, w^n\, e^{-\pi(w\overline{w})}$$

since we know the Fourier transform of a Gaussian. A similar computation with roles of $z, \overline{z}$ reversed accomplishes the other computation. That is, $(x \pm iy)^n e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-|n|}$. ///

[13.2] Let S, T be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for V consisting of simultaneous eigenfunctions for S, T.

Discussion: The Hilbert space V is the closure of the orthogonal direct sum of eigenspaces V_λ for T. For $\lambda \neq 0$, V_λ is finite-dimensional, so is necessarily closed, and V_0 is the orthogonal complement of the sum of all other eigenspaces, so is closed. Since $ST = TS$, we find that S stabilizes each V_λ:

$$T(Sv) \;=\; (TS)v \;=\; (ST)v \;=\; S(Tv) \;=\; S(\lambda v) \;=\; \lambda \cdot Sv \qquad \text{(for all } v \in V_\lambda)$$

[0.1] Claim: The restriction of a compact operator to a closed subspace $W \subset V$ stabilized by it is still compact.

Proof: With B' the closed unit ball of W and B the closed unit ball of V, $TB' \subset TB$. Using the total-boundedness criterion for precompactness, given $\varepsilon > 0$, TB is covered by finitely-many ε-balls $v_i + B_\varepsilon$. Among the intersections $W \cap (v_i + B_\varepsilon)$, the non-empty ones are open balls of radius at most ε. Thus, TB' is a precompact set, and $T|_W$ is a compact operator. ///

Thus, S is a compact operator on each V_λ, so evey V_λ has an orthonormal basis of S-eigenvectors. These are also λ-eigenvectors for T, so they are simultaneous eigenvectors. ///

[13.3] Show that $\varphi \to \int_{\mathbb{R}} e^{x^2}\, \varphi(x)\, dx$ is a distribution.

Discussion: From the characterization of the topology on $\mathcal{D}$, a compatible family of continuous linear functionals λ_K on the Fréchet spaces

$$\mathcal{D}_K \;=\; \{f \in \mathcal{D} : \operatorname{spt} f \subset K\}$$

uniquely determines a continuous linear function on $\mathcal{D}$. The *compatibility* condition is that, for $K \subset K'$, with inclusion map $i_K^{K'} : \mathcal{D}_{K'} \to \mathcal{D}_K$, we have $\lambda_{K'} \circ i_K^{K'} = \lambda_K$. For $\varphi \in \mathcal{D}_K$,

$$\Big|\int_{\mathbb{R}} e^{x^2}\, \varphi(x)\, dx\Big| \;\le\; \Big(\sup_{x\in K} e^{x^2}\Big)\cdot \int_K |\varphi(x)|\, dx \;\le\; \Big(\sup_{x\in K} e^{x^2}\Big)\cdot \operatorname{meas}(K)\cdot \sup_K |\varphi(x)|$$

Since $\sup_K |\varphi(x)|$ is one of the seminorms defining the Fréchet space topology on $\mathcal{D}_K$, this proves the continuity of that functional on $\mathcal{D}_K$. *Compatibility* is clear, although the constants appearing in the corresponding estimate depend on the compacts. ///

[13.4] Show that $\varphi \to \sum_{0\le n\in\mathbb{Z}} \varphi^{(n)}(n)$ is a distribution.

Discussion: As in the previous example, and since *compatibility* of the functionals on $\mathcal{D}_K$ is again clear, it suffices to prove that the indicated function is continuous on every $\mathcal{D}_K$. For $\varphi \in \mathcal{D}_K$,

$$\sum_{0\le n\in\mathbb{Z}} \varphi^{(n)}(n) \;=\; \sum_{0\le n\in\mathbb{Z}\cap K} \varphi^{(n)}(n)$$

which is a *finite* linear combination of *translates* of *derivatives* of δ. A finite linear combination of translates of derivatives of a distribution is again a distribution, and restriction to $\mathcal{D}_K$ gives a continuous linear functional on $\mathcal{D}_K$, since also $\mathcal{D}_K \to \mathcal{D}$ is continuous. Thus, the original functional is continuous on $\mathcal{D}$, and thus is a distribution. ///

[13.5] (Without invoking classification of distributions supported at a point) show that $\varphi \to \sum_{0\le n\in\mathbb{Z}} \varphi^{(n)}(0)$ is *not* a distribution.

Discussion: Since $\mathcal{D}_K \to \mathcal{D}$ is continuous, if the indicated map gave a distribution, it would give a continuous linear functional on every $\mathcal{D}_K$. We will show that this apparent functional does not give a continuous linear functional on any $\mathcal{D}_K$ where $K \ni 0$. It suffices to exhibit a test function φ such that the finite partial sums $u_N(\varphi) = \sum_{n\le N} \varphi^{(n)}(0)$ do not converge.

The function $f(x) = \sum_{j\ge 0} \frac{2^j\, x^j}{j!}$ is smooth on R. Let φ_o be a test function that is identically 1 on a neighborhood of 0. Then $\varphi(x) = \varphi_o(x)\cdot f(x)$ is a test function whose derivatives evaluated at 0 are those of f, namely, 2^j. Then $u_N(\varphi) = 1+2+4+8+\ldots+2^N$. The sequence of these goes to $+\infty$. ///

[0.2] Remark: In fact, $\varphi \to \sum_{n\ge 0} \frac{\varphi^{(n)}(0)}{n!}$ is not a distribution, and, further, for *no* sequence $c_1, c_2, \ldots$ of non-zero numbers (no matter how rapidly decreasing) is $\varphi \to \sum_{n\ge 0} c_n \varphi^{(n)}(0)$ a distribution. We certainly know this from the classification, on one hand. On another hand, a theorem of E. Borel asserts that, for any sequence b_n of complex numbers, there is a smooth function whose Taylor coefficients at 0 are the given sequence of numbers. In particular, this applies to $b_n = e^{e^n}$ or any other sequence, no matter how rapidly increasing.

[13.6] Let T be a continuous/bounded self-adjoint operator on a Hilbert space V, with spectrum consisting of just two points $\lambda \ne \mu$. Show that the isomorphism $C^o(\{\lambda,\mu\}) \approx \overline{\mathbb{R}[T]}$ implies that V is the direct sum of λ and μ eigenspaces.

Discussion: Let f, g be continuous real-valued functions such that $f(\lambda) = 1$, $f(\mu) = 0$, $g(\lambda) = 0$, and $g(\mu) = 1$. On the spectrum of T, $(x-\lambda)f(x) = 0$, so $(T-\lambda)f(T) = 0$, so T acts by λ on the image $f(T)V$ of $f(T)$. Similarly, T acts by μ on the image $g(T)V$ of g. On the spectrum of T, $f^2(x) = f(x)$, since both are 1 on λ and 0 on μ. Similarly, $g^2(x) = g(x)$ on the spectrum. Since $f(T)$ and $g(T)$ are in the operator-norm closure of $\mathbb{R}[T]$, they are self-adjoint. Self-adjoint idempotent operators are orthogonal projectors, so $f(T)$ is a (non-zero, because f restricted to the spectrum is not identically 0) orthogonal projector to *some* (non-zero) subspace of the λ-eigenspace, and similarly for $g(T)$.

Since $f + g = 1$ on the spectrum of T, $f(T)$ must map to the *whole* λ-eigenspace, and similarly for g, and the (orthogonal) sum of these eigenspaces is the whole Hilbert space. ///

[0.3] Remark: One should prove that an operator-norm limit of self-adjoint functions is self-adjoint.
