(*September 14, 2016*)

Review examples 01

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-01.pdf]

If you want feedback on your write-ups on any of these examples, please get your write-ups by Monday, 19 Sept, 2016.

[01.1] Prove (or review the proof) that a *uniform* pointwise limit of continuous, real-valued functions on $[a, b]$ is continuous.

[01.2] Prove (or review the proof) of the *Fundamental Theorem of Calculus*: for a *continuous* function f on $[a, b]$, the function $F(x) = \int_a^x f(t)\,dt$ is *continuously differentiable*, and has derivative f. (Use Riemann's integral.)

[01.3] Prove (or review the proof) that for a sequence of real-valued functions f_n on $[0, 1]$ approaching f *uniformly* pointwise, $\lim_n \int_0^1 f_n(x)\,dx = \int_0^1 \lim_n f_n(x)\,dx$. (Use Riemann's integral.)

[01.4] Show that every open subset of $\mathbb{R}$ is a *countable* union of open intervals.

[01.5] Define an *(outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ given by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.

[01.6] Give an example of a sequence of continuous real-valued functions $\{f_n\}$ on $[0, 1]$ whose pointwise limit $f(x) = \lim_n f_n(x)$ is $1/q$ on rational numbers p/q in lowest terms, and is 0 for x irrational.

The following examples are essentially irrelevant to us, but have some entertainment value, and would once have been considered significant. But by now, if anything, such examples and counter-examples tend to illustrate the futility of otherwise-appealing enterprises.

[01.7] (*) Given an enumeration $r_1, r_2, \ldots$ of rational numbers in $[0, 1]$, and given a sequence $y_1, y_2, \ldots \to 0$ of real numbers going to 0, construct a sequence of continuous real-valued functions $\{f_n\}$ on $[0, 1]$ whose pointwise limit $f(x) = \lim_n f_n(x)$ is y_n on r_n, and is 0 for x irrational.

[01.8] (*) Use this to construct a sequence $\{\{f_{mn} : n = 1, 2, \ldots\} : m = 1, 2, \ldots\}$ of sequences such that $\lim_m(\lim_n f_{mn}(x)) = 1$ for rational x, and 0 for irrational x.

[01.9] (**) Show that for any sequence $\{f_n\}$ of real-valued functions on $[0, 1]$ with $0 \le f_n(x) \le 1$ for all x, n, and with $f_n(x) \to 1$ for rational x, there are uncountably-many $y \in [0, 1]$ with $\limsup f_n(y) = 1$.
