

(October 4, 2016)

Examples 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-ex-02.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 03 Oct, 2016.

[02.1] Prove that every open in \mathbb{R}^2 is a countable union of Cartesian products $(a, b) \times (c, d)$ of open intervals.

[02.2] Define tent functions of width w and height h centered at 0 by

$$t_{w,h}(x) = \begin{cases} 0 & (\text{for } x \leq -w) \\ \frac{h}{w} \cdot (x + w) & (\text{for } -w \leq x \leq 0) \\ h - \frac{h}{w} \cdot x & (\text{for } 0 \leq x \leq w) \\ 0 & (\text{for } x \geq w) \end{cases}$$

Show that the functions $f_n(x) = t_{\frac{1}{n},n}(x - \frac{1}{n})$, a sequence of narrowing tents just to the right of 0, go to 0 *pointwise* (everywhere!), but that

$$\lim_n \int_{\mathbb{R}} f_n(x) \cdot g(x) dx = g(0) \quad (\text{for all } g \in C^o(\mathbb{R}))$$

[02.3] Show that the functions $f_n(x) = t_{\frac{1}{n},n^2}(x - \frac{1}{n}) - t_{\frac{1}{n},n^2}(x + \frac{1}{n})$, whose graphs are tall tents of area n upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

$$\lim_n \int_{\mathbb{R}} f_n(x) \cdot g(x) dx = 2 \cdot g'(0) \quad (\text{for differentiable } g \text{ with derivative } g' \text{ in } C^o(\mathbb{R}))$$

[02.4] Show that the closed unit ball in ℓ^2 , although *closed* and *bounded*, is *not compact*, by showing it is not *sequentially* compact.

[02.5] Show that the *Hilbert cube*

$$C = \{(z_1, z_2, \dots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals ε_n ,

$$C(\varepsilon) = \{(z_1, z_2, \dots) \in \ell^2 : |z_n| \leq \varepsilon_n\}$$

is compact if and only if $\sum_n |\varepsilon_n|^2 < \infty$.

[02.6] Show that a closed interval $[a, b]$ has the expected Lebesgue measure, namely, $|b - a|$, by showing that the *inf* of $\sum_{j=1}^n |b_n - a_n|$ for all finite open covers $[a, b] \subset \bigcup_{j=1}^n (a_j, b_j)$ is $|b - a|$.

[02.7] Let f be a continuous function on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. Show that $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive Lebesgue measure.

[02.8] Analogous to the Cantor middle-thirds set, but with shrinking ratios of what's removed, form a subset of $[0, 1]$ by first removing the middle 1/4 of $[0, 1]$, then the middle 1/9s of the remaining intervals, then the middle 1/16's of the remaining intervals, then the middle 1/25's of the remaining intervals, ... Show that the nested intersection of these sets has *positive* measure. (In fact, its measure is $1/2\pi$.)