(*October 31, 2016*)

Examples 04

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 28 Oct 2016.

[04.1] Comparing L^p spaces. Let $1 \le p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

[04.2] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

[04.3] In ℓ^2, show that the point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

[04.4] For a measurable set $E \subset [0, 2\pi]$, show that

$$\lim_{n\to\infty} \int_E \cos nx\, dx \;=\; 0 \;=\; \lim_{n\to\infty} \int_E \sin nx\, dx$$

[04.5] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. Write out the conclusion of Parseval's theorem for this function.

[04.6] For fixed $y \in [0, 2\pi]$, show that there is *no* $f_y \in L^2[0, 2\pi]$ so that $\langle g, f_y\rangle = g(y)$ for all $g \in L^2[0, 2\pi]$.

[04.7] (In contrast to the previous example's outcome.) Let V be the complex vector space of power series $f(z) = \sum_{n\ge 0} c_n\, z^n$ convergent on the open unit disk D in $\mathbb{C}$, having finite norm

$$|f| \;=\; \Big(\int_D |f(x+iy)|^2\, dx\, dy\Big)^{\frac{1}{2}}$$

with hermitian inner product

$$\langle f, g\rangle \;=\; \int_D f(x+iy) \cdot \overline{g(x+iy)}\, dx\, dy$$

Show that $\langle z^m, z^n\rangle = 0$ unless $m = n$, in which case it is $\frac{\pi}{n+1}$, and that $\psi_n(z) = z^n \cdot \frac{\sqrt{n+1}}{\sqrt{\pi}}$ is an orthonormal basis for V. Show that the sum $f_w(z) = \sum_{n\ge 0} \psi_n(z)\, \overline{\psi_n(w)}$ converges absolutely for $z, w \in D$, and that

$$\langle g(-),\, f_w\rangle \;=\; g(w) \qquad \text{(for } w \text{ in the disk)}$$

Show that for each fixed $w \in D$, pointwise evaluation $g \to g(w)$ is a continuous linear functional on V.
