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Examples 05

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-05.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 06 Nov 2016.

[05.1] Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) *closed*.

[05.2] For a subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the closure of the subspace W in V.

[05.3] Let $T : \ell^2 \to \ell^2$ be the *right shift*: $T(z_1, z_2, z_3, \ldots) = (0, z_1, z_2, z_3, \ldots)$. Determine the *adjoint* T^*.

[05.4] Show that for $0 < x < 1$

$$\sum_{n\ge 1} \frac{\sin 2\pi nx}{n} \;=\; \pi(\tfrac{1}{2} - x)$$

[05.5] Prove that every $f \in C_c^o(\mathbb{R})$ can be uniformly approximated (in sup norm) arbitrarily well as superpositions of Gaussians: given $\varepsilon > 0$, there is $\varphi \in C_c^o(\mathbb{R})$ and sufficiently large n such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi)\cdot ne^{-\pi n^2(\xi - x)^2}\, d\xi \Big| \;<\; \varepsilon$$

[05.6] Without worrying too much about identifying the finite, positive constant $\int_{\mathbb{R}} \frac{(\sin x)^2}{x^2}\, dx$, prove that, for given $f \in C_c^o(\mathbb{R})$, given $\varepsilon > 0$, there is sufficiently large n and a function $\varphi \in C_c^o(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi)\cdot \frac{(\sin n(x-\xi))^2}{(x-\xi)^2}\, d\xi \Big| \;<\; \varepsilon$$
