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# *Examples 06*

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2016-17/real-ex-06.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, 23 Nov 2016.

**[06.1]** Let $c_1, c_2, \ldots$ be positive real, converging monotonically to 0. For $0 < x < 1$, prove that $\sum_{n \geq 0} c_n \, e^{2\pi inx}$ *converges pointwise.*

**[06.2]** Show that the sup-norm completion of the space $C^o_c(\mathbb{R})$ of compactly-supported continuous functions is the space $C^o_o(\mathbb{R})$ of continuous functions going to 0 at infinity. An analogous assertion and argument should hold for any topological space in place of $\mathbb{R}$.

**[06.3]** Show that the translation action $T_x f(y) = f(y+x)$ on the Banach space $C^o_{bdd}(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ is *not* continuous. That is, $\mathbb{R} \times C^o_{bdd}(\mathbb{R}) \to C^o_{bdd}(\mathbb{R})$ by $x \times f \to T_x f$ is *not* continuous. In particular, find a particular $f \in C^o_{bdd}(\mathbb{R})$ with $|f|_{C^o} = 1$ such that, there is a sequence $\delta_n \to 0$ of non-zero numbers $\delta_n$ such that $|T_{\delta_n} f - f|_{C^o} = 1$.

**[06.4]** Prove that the *Volterra operator* $Vf(x) = \displaystyle\int_0^x f(t) \, dt$ on $C^o[0,1]$ or on $L^2[0,1]$ has no (not-identically-zero) eigenvalues/eigenvectors.

**[06.5]** Let $K(x,y) = |x-y|$, and let

$$Tf(x) \;=\; \int_a^b K(x,y) \, f(y) \, dy \qquad\qquad (\text{for } f \in L^2[a,b])$$

Find some eigenvalues/eigenfunctions for the operator $T$. (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

**[06.6]** Show that the *principal value* functional

$$f \;\longrightarrow\; PV \int_{\mathbb{R}} \frac{f(x)}{x} \, dx \;=\; \lim_{\varepsilon \to 0} \Big( \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x} \, dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x} \, dx \Big)$$

is equal to

$$-\int_{\mathbb{R}} f'(x) \cdot \log|x| \, dx$$

for $f$ *continuously* differentiable near 0, with $f \in L^2(\mathbb{R})$, $f(x) \to 0$ as $|x| \to \infty$, and $|f'(x)| \ll \frac{1}{1+x^2}$ for easy convergence.