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Examples 07

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-07.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 05 Dec 2016.

[07.1] Show that the direct sum $X \oplus Y$ of two Hilbert spaces (the underlying set is the collection of ordered pairs (x, y) with $x \in X$ and $y \in Y$, also denoted $x \oplus y$) with

$$\langle x \oplus y,\ x' \oplus y' \rangle \;=\; \langle x, x' \rangle_X + \langle y, y' \rangle_Y \qquad \text{(for } x, x' \in X \text{ and } y, y' \in Y\text{)}$$

is a Hilbert space.

[07.2] Show that the direct sum $X \oplus Y$ of two Banach spaces with

$$|x \oplus y| \;=\; |x|_X + |y|_Y \qquad \text{(for } x \in X \text{ and } y \in Y\text{)}$$

is a Banach space. Note that this produces a different outcome than the previous example when X, Y are Hilbert spaces. To encompass both examples at once, show that for any $p \geq 1$

$$|x \oplus y|_p \;=\; \Big(|x|_X^p + |y|_Y^p\Big)^{1/p} \qquad \text{(for } x \in X \text{ and } y \in Y\text{)}$$

makes $X \oplus Y$ a Banach space.

[07.3] Let $\psi_n(x) = e^{2\pi i n x}$. Let $\delta_{\mathbb{Z}}$ be the *Dirac comb*, that is, a periodic version of Dirac's δ, describable as having Fourier series

$$\delta_{\mathbb{Z}} \;=\; \sum_{n \in \mathbb{Z}} 1 \cdot \psi_n \qquad \text{(converging in } H^{-1}(\mathbb{T}) \text{ or even } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon > 0\text{)}$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda \cdot u \;=\; \delta_{\mathbb{Z}}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division*. Show that the equation $v'' - \lambda v = f$ is solved by

$$v \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_0^1 u(x-t)\, f(t)\, dt$$

[07.4] Let V be a vector space, with norms $|\cdot|_1$ and $|\cdot|_2$. Suppose that $|v|_2 \geq |v|_1$ for all $v \in V$. Show that the identity map $i : V \to V$ is continuous, where the source is given the $|\cdot|_2$ topology and the target is given the $|\cdot|_1$ topology. Show that if a sequence $\{v_n\}$ in V is $|\cdot|_2$ Cauchy, then it is $|\cdot|_1$-Cauchy. Let V_j be the completion of V with respect to the metric $|v - v'|_j$. Show that we can *extend i by continuity* to a continuous linear map $I : V_2 \to V_1$, that is, by

$$I(V_2\text{-limit of } V_2\text{-Cauchy sequence } \{v_n\}) \;=\; V_1\text{-limit of } \{v_n\}$$

[07.5] Let X, Y be Hilbert spaces over $\mathbb{R}$, to avoid the distraction of complex conjugation. Let $i_X : X \to X^*$ be the Riesz-Fréchet isomorphism of X to its dual X^*, by $x \to \langle -, x \rangle$. For a continuous linear $T : X \to Y$, let $T^* : Y^* \to X^*$ be the *adjoint*, defined as usual by $(T^*\mu)(x) = \mu(Tx)$ for $\mu \in Y^*$ and $x \in X$. Perhaps surprisingly, the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & Y \\ {\scriptstyle i_X}\downarrow & & \downarrow{\scriptstyle i_Y} \\ X^* & \xleftarrow{T^*} & Y^* \end{array}$$

does not generally *commute*. Give an example of failure. (*Hint:* It already suffices to take $X = Y = \mathbb{R}$. In fact, the diagram commutes *if and only* if $T : X \to Y$ is an *isometry* to the image $T(X)$.)

[07.6] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2 \, dx$. (*Hint:* use Plancherel.)

[07.7] Show that for any $f \in C^o_c(\mathbb{R})$, $\displaystyle\lim_{|t|\to+\infty} \int_{\mathbb{R}} e^{itx} f(x)\, dx = 0$.
