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Examples 08

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-08.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, Feb 01, 2017.

[08.1] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\, dx \Big| \;<\; C \cdot \sqrt{\delta}$$

Formulate and prove the corresponding assertion for L^p with $1 < p < \infty$.

[08.2] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\Big| \int_{t-\delta}^{t+\delta} f(x)\, dx \Big| \;<\; \varepsilon$$

Sharpen the first example to show that

$$\int_{t-\delta}^{t+\delta} f(x)\, dx \;=\; o(\sqrt{\delta}) \qquad\qquad (\text{as } \delta \to 0^+)$$

where Landau's little-o notation is that $f(x) = o(g(x))$ as $x \to a$ when $\lim_{x\to a} f(x)/g(x) = 0$.

[08.3] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you assert: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$.)

[08.4] Let $K(x,y) \in L^2([a,b]\times[a,b])$, and attempt to define a map $T : L^2[a,b] \to L^2[a,b]$ by

$$Tf(x) \;=\; \int_a^b K(x,y)\, f(y)\, dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$|Tf|_{L^2[a,b]} \;\le\; |K|_{L^2([a,b]\times[a,b])} \cdot |f|_{L^2[a,b]}$$

(One would say that $K(,)$ is a *Schwartz kernel* for the map T. Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y,x)}$.

[08.5] The Volterra operator $Vf(x) = \int_0^x f(y)\, dy$ on $L^2[0,1]$ has kernel

$$K(x,y) \;=\; \begin{cases} 1 & (\text{for } 0 \le y \le x \le 1) \\ \\ 0 & (\text{for } 0 \le x < y \le 1) \end{cases}$$

Determine the (Schwartz) kernel for $T = V \circ V^*$. Find some eigenfunctions for T. (Recall that V has no eigenfunctions!) (*Hint:* apply d/dx to the equation $Tf = \lambda \cdot f$ and presume that the differentiation passes inside the integral.)

[08.6] The sawtooth function is first defined on $[0,1)$ by $\sigma(x) = x - \frac{1}{2}$, and then extended to $\mathbb{R}$ by periodicity so that $\sigma(x+n) = \sigma(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. Determine its Fourier expansion. Describe the derivative σ' of σ.

[08.7] Given f in the Schwartz space $\mathscr{S}$, show that there is $F \in \mathscr{S}$ with $F' = f$ if and only if $\int_{\mathbb{R}} f = 0$.

[08.8] Show that the evaluation functional (Dirac delta) $\delta_{x_o} : f \to f(x_o)$ is a continuous linear functional on $\mathscr{S}$, for every $x_o \in \mathbb{R}$. Determine the Fourier transform of δ_{x_o}.

[08.9] Let $u(x) = e^x \cdot \sin(e^x)$. Explain in what sense the integral $\displaystyle\int_{\mathbb{R}} f(x)\, u(x)\, dx$ converges for every $f \in \mathscr{S}$.
