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Examples 09

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-09.pdf]

For feedback on these examples, please get your write-ups to me by Monday, Feb 13, 2017.

[09.1] Give a *persuasive* proof that the function

$$f(x) = \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this kind of construction to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0,1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1,2,3,\ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n,n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1), -n]$ and monotonically from 1 to 0 in $[n, n+1]$.

[09.2] For a family of smooth cut-off functions $\{f_n\}$ as in the previous example, and for a smooth function g, let $g_n(x) = f_n \cdot g$. Observe that every g_n is a test function. Show that $g_n \to g$ in the $\mathcal{D}^*$-topology.

[09.3] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$. (Since $\mathscr{S}$ is strictly larger than $\mathcal{D}$, this implies that $\sin(nx) \to 0$ in the $\mathcal{D}^*$-topology.)

[09.4] Show that $e^{-\varepsilon\pi x^2} \to 1$ as $\varepsilon \to 0^+$ in the $\mathscr{S}^*$ topology. Compute the Fourier transforms of the functions $e^{-\varepsilon\pi x^2}$, and show that they go to δ in the $\mathscr{S}^*$ topology. Obtain the corollary that $\widehat{1} = \delta$ (extended Fourier transform).

[09.5] Let $-\infty < a < b < c < +\infty$, and

$$f(x) = \begin{cases} 0 & (\text{for } x < a) \\ A & (\text{for } a < x < b) \\ B & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Show that (extended) $\frac{d}{dx} f = A\delta_a + (B-A)\delta_b - B\delta_c$.

[09.6] Let $-\infty < a < b < c < +\infty$, and

$$f(x) = \begin{cases} 0 & (\text{for } x < a) \\ A_1 x + A_2 & (\text{for } a < x < b) \\ B_1 x + B_2 & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Compute (extended) $\frac{d^2}{dx^2} f$. (It is a linear combination of $\delta_a, \delta_a', \delta_b, \delta_b', \delta_c, \delta_c'$.)
