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Examples 10

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-10.pdf]

For feedback on these examples, please get your write-ups to me by Monday, Mar 06, 2017.

[10.1] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, first for $f \in \mathscr{S}(\mathbb{R}^n)$, then for $f \in \mathscr{S}(\mathbb{R}^n)^*$.

[10.2] Compute $\widehat{\cos x}$.

[10.3] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\mathrm{spt}\,\varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x \cdot u = 0$ implies that u is a multiple of δ itself.

[10.4] Show that the principal value functional $u(\varphi) = P.V. \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$ satisfies $x \cdot u = 1$.

[10.5] Compute the Fourier transform of the sign function

$$\mathrm{sgn}(x) \;=\; \begin{cases} 1 & (\text{for } x > 0) \\ \\ -1 & (\text{for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx}\mathrm{sgn} = 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi i x$, this implies that $(2\pi i)x \cdot \widehat{\mathrm{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x \cdot \widehat{\mathrm{sgn}} = 1$.

[10.6] Compute the Fourier transform of $|x|$.

[10.7] Determine the Schwartz kernel $K(,)$ for the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$, and show that it is in $H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n})$ for every $\varepsilon > 0$.
