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Examples 11

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[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-11.pdf]

For feedback on these examples, please get your write-ups to me by Monday, Mar 20, 2017.

[11.1] For $T : V \to V$ a continuous (=bounded) linear map of a Banach space V to itself, show that the operator norm is an upper bound for absolute values of all eigenvalues λ: $|\lambda|_{\mathbb{C}} \le |T|_{\text{op}}$. Further, show that $|T|_{\text{op}}$ is an upper bound for *all* of the spectrum, that is, $T - \lambda$ is invertible for $|\lambda|_{\mathbb{C}} > |T|_{\text{op}}$.

[11.2] *(Approximate eigenvectors and continuous spectrum)* Let $T : V \to V$ be a continuous linear operator on a Hilbert space V. For $\lambda \in \mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T - \lambda)v_n \to 0$ as $n \to +\infty$ is an *approximate eigenvector* for λ. Show that for λ *not* an eigenvalue for T, λ has an approximate eigenvector if and only if λ is in the spectrum of T.

[11.3] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x) \cdot \sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectum and continuous spectrum, that is, no left-over *residual* spectrum.)

[11.4] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0, 1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0, 1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

[11.5] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots)$. Show that T is *compact* if and only if $r_n \to 0$.

[11.6] Let V be the Volterra operator $Vf(x) = \int_0^x f(t)\,dt$ on $L^2[0, 1]$. Show that $|V^n|_{\text{op}}^{1/n} \to 0$ as $n \to +\infty$. Show that the spectrum of V is just $\{0\}$.
