(*March 31, 2017*)

Examples 12

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2016-17/real-ex-12.pdf]

For feedback on these examples, please get your write-ups to me by Monday, April 10, 2017.

[12.1] Let T be a compact operator $T : V \to W$ for Hilbert spaces V, W. For S a continuous/bounded operator on V, show that $T \circ S : V \to W$ is compact. For R a continuous/bounded operator on W, show that $R \circ T : V \to W$ is compact.

[12.2] (*Rellich's lemma on the circle*) For $s < t \in \mathbb{R}$, show that the inclusion map $H^t(\mathbb{T}) \to H^s(\mathbb{T})$ is *compact*. (*Hint:* Use the orthogonal bases $\psi_n(x) = e^{2\pi i n x}$, and note that their lengths in $H^s(\mathbb{T})$ vary depending on s. Thus, if we choose isomorphisms of H^s and H^t to $\ell^2(\mathbb{Z})$, the inclusion $H^t \to H^s$ sending $\psi_n \to \psi_n$ will *not* be the identity map on those copies of $\ell^2(\mathbb{Z})$.)

[12.3] Let $K(,)$ be a measurable function on $\mathbb{R}^2$, with a bound B such that $\int_{\mathbb{R}} |K(x,y)|\,dx \le B$ for every y, and $\int_{\mathbb{R}} |K(x,y)|\,dy \le B$ for every x. Show that $Tf(x) = \int_{\mathbb{R}} K(x,y)\,f(y)\,dy$ gives a continuous linear map $L^p \to L^p$ for every $1 < p < \infty$, with $|Tf|_{L^p} \le B \cdot |f|_{L^p}$. (*Hint:* Hölder's inequality.)

[12.4] (A simple instance of *Young's inequality*) In the previous example, let $K(x,y) = k(x-y)$ for $k \in L^1(\mathbb{R})$, so that $Tf(x) = (k * f)(x)$. Show that $|Tf|_{L^p} \le |k|_{L^1} \cdot |f|_{L^p}$.

[12.5] Solve $-u'' + u = \delta$ on $\mathbb{R}$. (*Hint:* use Fourier transform. Knowing how to evaluate standard/iconic integrals by residues would be convenient, but/and the relevant integral was done in an earlier example-discussion.)

[12.6] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution. (And reflect on the Fredholm alternative?)

[12.7] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1. (And reflect on the Fredholm alternative?)
