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Review examples discussion 00

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[This document is http://www.math.umn.edu/˜garrett/m/real/notes_2017-18/real-disc-00.pdf]

[00.1] (There is not much hope in making sense of the outcome of an uncountable number of non-zero operations:) Let Ω be an *uncountable* collection of positive real numbers. Letting F range over all finite subsets of Ω, show that $\sup_F \sum_{\alpha\in F} \alpha = +\infty$.

Discussion: Let $\Omega_1 = \{\omega \in \Omega : \omega > 1\}$, and for $n = 2, 3, \ldots$, let $\Omega_n = \{\omega \in \Omega : \frac{1}{n} < \omega \le \frac{1}{n-1}\}$. There are countably many such sets, so in (at least) one of them Ω_{n_o} there must be infinitely-many elements of Ω (or else Ω would be a countable union of countable sets, hence countable). Then

$$\sup_F \sum_{\alpha\in F} \;\ge\; \sup_{F\subset\Omega_{n_o}} \sum_{\alpha\in F} \;\ge\; \sup_{F\subset\Omega_{n_o}} \#F\cdot\frac{1}{n_o} \;=\; \frac{1}{n_o}\sup_{F\subset\Omega_{n_o}} \#F \;=\; +\infty$$

because Ω_{n_o} is infinite. ///

[00.2] Prove (or review the proof) that a continuous real-valued function f on a finite closed interval $[a, b] \subset \mathbb{R}$ is *uniformly* continuous: for all $\varepsilon > 0$ there is $\delta > 0$ such that, for all $x, y \in [a, b]$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$.

Discussion: Given $\varepsilon > 0$ and $x \in [a, b]$, take $\delta_x > 0$ such that $|x' - x| < 2\delta_x$ implies $|f(x') - f(x)| < \varepsilon/2$. The open intervals $(x - \delta_x, x + \delta_x)$ cover the compact set $[a, b]$, so there is a finite subcover $\{(x_j - \delta_{x_j}, x_j + \delta_{x_j}) : j = 1, \ldots, N\}$. The minimum $\delta = \min_{j=1,\ldots,N} \delta_j$ is positive (see above). For given $x \in [a, b]$, $x \in (x_j - \delta_{x_j}, x_j + \delta_{x_j})$ for some j.

For x' such that $|x' - x| < \delta$, we have $|x' - x_j| \le |x' - x| + |x - x_j| \le \delta + \delta_j \le 2\delta_j$, so $|f(x') - f(x_j)| < \varepsilon/2$, and

$$|f(x') - f(x)| \;\le\; |f(x') - f(x_j)| + |f(x_j) - f(x)| \;<\; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \;=\; \varepsilon$$

which is the uniform continuity. ///

[00.3] Prove (or review the proof) that a *uniform* pointwise limit of continuous, real-valued functions on $[a, b]$ is continuous.

Discussion: This is the archetype of a three-ε argument. Let the sequence by $\{f_n\}$, and the pointwise limit $f(x) = \lim_n f_n(x)$. Given $\varepsilon > 0$, by the *uniform* pointwise approach to the limit, take n_o large enough so that for all $m, n \ge n_o$, for all $x \in [a, b]$, $|f_m(x) - f_n(x)| < \varepsilon$. Then $|f(x) - f_n(x)| \le \varepsilon$ for all $x \in [a, b]$, for all $n \ge n_o$. By the uniform continuity of f_{n_o} on $[a, b]$, let $\delta > 0$ so that $|f_{n_o}(x) - f_{n_o}(y)| < \varepsilon$ for all $|x - y| < \delta$. Then

$$|f(x) - f(y)| \;\le\; |f(x) - f_{n_o}(x)| + |f_{n_o}(x) - f_{n_o}(y)| + |f_{n_o}(y) - f(y)| \;<\; \varepsilon + \varepsilon + \varepsilon$$

as desired. ///

Note: In the latter situation, there is no compulsion to go back and replace ε by $\varepsilon/3$, since it is obviously possible to do so.

[00.4] Prove (or review the proof) of the *Fundamental Theorem of Calculus*: for a *continuous* function f on $[a, b]$, the function $F(x) = \int_a^x f(t)\,dt$ is *continuously differentiable*, and has derivative f. (Use Riemann's integral.)

Discussion: We use the finite additivity property

$$\int_a^c f(x)\,dx \;=\; \int_a^v f(x)\,dx + \int_v^c f(x)\,dx \qquad \text{(for all } v < c \text{ between } a \text{ and } b\text{)}$$

Thus,

$$\frac{F(x+\delta)-F(x)}{\delta} - f(x) \;=\; \frac{\int_x^{x+\delta} f(t)\,dt}{\delta} - f(x)$$

By continuity of f, given $\varepsilon > 0$, take $\delta_o > 0$ sufficiently small so that

$$\sup_{y:x\le y\le x+\delta_o} |f(y)-f(x)| \;<\; \varepsilon$$

Then

$$\frac{\int_x^{x+\delta} f(t)\,dt}{\delta} - f(x) \;<\; \frac{(f(x)+\varepsilon)\cdot\delta}{\delta} - f(x) \;=\; \varepsilon$$

and, similarly,

$$\frac{\int_x^{x+\delta} f(t)\,dt}{\delta} - f(x) \;>\; \frac{(f(x)-\varepsilon)\cdot\delta}{\delta} - f(x) \;=\; -\varepsilon$$

Thus, given $\varepsilon > 0$, there is $\delta_o > 0$ such that for every $0 < \delta \le \delta_o$

$$\Big|\frac{F(x+\delta)-F(x)}{\delta} - f(x)\Big| \;<\; \varepsilon$$

(Finding $\delta_o < 0$ for the same inequality is similar.) ///

[The following is a discussion of the question I **meant** *to ask!!!]*

[00.5] Prove (or review the proof) that for a sequence of real-valued functions f_n on $[0,1]$ approaching f *uniformly* pointwise, $\lim_n \int_0^1 f_n(x)\,dx = \int_0^1 \lim_n f_n(x)\,dx$. (Use Riemann's integral.)

Discussion: Given $\varepsilon > 0$, let n_o be large enough so that for all $n \ge n_o$, for all $x \in [a,b]$, $|f_n(x) - f(x)| < \varepsilon$. Using *linearity* of integrals,

$$\int_a^b f(x)\,dx \;=\; \int_a^b f(x) - f_{n_o}(x)\,dx + \int_a^b f_{n_o}(x)\,dx$$

Upper and lower bounds are obtained from any upper and lower Riemann sums, for any partition $a = x_1 < \ldots < x_n = b$ of the interval:

$$\int_a^b f(x) - f_{n_o}(x)\,dx \;<\; \sum_{j=1}^{n}(x_{j+1}-x_j)\cdot\varepsilon \;=\; (b-a)\cdot\varepsilon$$

and similarly for a lower bound. ///

[00.6] Show that every open subset of $\mathbb{R}$ is a *countable* union of open intervals.

Discussion: Let S be the set. For $s \in S$, since S is open, there is $0 < \delta_s \in \mathbb{Q}$ such that $(s-2\delta_s, s+2\delta_s) \subset S$. By density of $\mathbb{Q}$ in $\mathbb{R}$ there is q_s in the smaller interval $(s-\delta_s, s+\delta_s)$. Certainly $s \in (q_s - \delta_s, q_s + \delta_s)$, and $(q_s - \delta_s, q_s + \delta_s) \subset S$, because for $|t - q_s| < \delta_s$

$$|s-t| \;\le\; |s-q_s| + |q_s - t| \;<\; \delta + \delta$$

The collection of *all* pairs $(q, \delta) \in \mathbb{Q} \times \mathbb{Q}$ of rationals q, δ is countable, so the subset of (distinct) pairs occuring as q_s, δ_s for $s \in S$ is countable. (Apparently many of the pairs (q, δ) appear as (q_s, δ_s) for many different $s \in S$.) ///

[00.7] Define *Lebesgue (outer) measure* $\mu(E)$ of subsets E of $\mathbb{R}$ given by

$$\mu(E) \;=\; \inf\{\sum_{n=1}^{\infty} |b_n - a_n| \;:\; E \subset \bigcup_{n=1}^{\infty} (a_n, b_n)\}$$

Show that $\mu(\mathbb{Q}) = 0$. Show that $\mu(M) = 0$, where M is Cantor's middle-thirds set.

Discussion: Enumerate the rationals as $r_1, r_2, \ldots$. Given $\varepsilon > 0$, let $U_{n,\varepsilon}$ be the interval $(r_n - \frac{\varepsilon}{2^n}, r + \frac{\varepsilon}{2^n})$. The union of these intervals contains $\mathbb{Q}$, and the sum of lengths is $\varepsilon \cdot (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = \varepsilon$.

The Cantor middle-thirds set can be described in terms of base-three expansions, as follows. All real numbers r in $[0,1]$ have (ternary) expansion $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0,1,2\}$. The expansion is unambiguous except for the possibility of coefficients all 2 beyond a certain point, which we exclude by using

$$\frac{2}{3^n} + \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \ldots \;=\; 2 \cdot \frac{3^{-n}}{1 - \frac{1}{3}} \;=\; 2 \cdot \frac{3^{1-n}}{3-1} \;=\; 3^{1-n}$$

Then the middle-thirds set C is the set of reals $r = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$ with all coefficients a_n in the set $\{0,2\}$ (with the convention excluding endlessly repeating 2's).

Alternatively, the middle-thirds set C is formed as a *nested intersection*, as follows. Let C_1 be $[0,1]$ with the middle third $(\frac{1}{3}, \frac{2}{3})$ removed. Let C_2 be C_1 with the middle third thirds $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$ removed, and so on. At each step, the sum of lengths of the remaining intervals is multiplied by $(1 - \frac{1}{3}) = \frac{2}{3}$, and the number of intervals is multiplied by 2. After n middle-third removals, the result C_n is a union of 2^n intervals each of length 3^{-n}. The Cantor middle-thirds set is $C = \bigcap_n C_n$.

Given $\varepsilon > 0$, choose n large enough so that $2^n/3^n < \varepsilon/2$. Cover each of the 2^n intervals of length 3^{-n} making up C_n by an open interval of length $2 \cdot 3^{-n}$. The sum of the lengths of these 2^n open intervals is

$$2^n \cdot (2 \cdot 3^{-n}) \;=\; 2 \cdot (2/3)^n \;<\; 2 \cdot \frac{\varepsilon}{2} \;=\; \varepsilon$$

This exhibits an open cover of C_n with sum of lengths less than ε. Since $C \subset C_n$, this gives such a cover of C itself, as desired. ///
