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Examples 01

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-01.pdf]

[01.1] Prove that every open in \mathbb{R}^2 is a countable union of Cartesian products $(a, b) \times (c, d)$ of open intervals.

Discussion: For a point u in (non-empty) open $U \subset \mathbb{R}^2$, there is an open ball about u of some positive radius and contained in U . Shrink the ball slightly to have positive *rational* radius r . Within distance $r/4$ of u there is a point $(x, y) \in \mathbb{Q}^2$. Then the open ball B_u of radius $r/4$ at (x, y) contains u . Thus, the square $(x - \frac{r}{2}, x + \frac{r}{2}) \times (y - \frac{r}{2}, y + \frac{r}{2})$ contains u . There are only countably-many possibilities for the intervals $(x - \frac{r}{2}, x + \frac{r}{2})$ and $(y - \frac{r}{2}, y + \frac{r}{2})$, so the set of such occurring for $u \in U$ is countable. ///

[01.2] Define tent functions of (half-) width w and height h centered at 0 by

$$t_{w,h}(x) = \begin{cases} 0 & (\text{for } x \leq -w) \\ \frac{h}{w} \cdot (x + w) & (\text{for } -w \leq x \leq 0) \\ h - \frac{h}{w} \cdot x & (\text{for } 0 \leq x \leq w) \\ 0 & (\text{for } x \geq w) \end{cases}$$

Show that the functions $f_n(x) = t_{\frac{1}{n},n}(x - \frac{1}{n})$, a sequence of narrowing tents just to the right of 0, go to 0 *pointwise* (everywhere!), but that

$$\lim_n \int_{\mathbb{R}} f_n(x) \cdot g(x) dx = g(0) \quad (\text{for all } g \in C^o(\mathbb{R}))$$

Discussion: Certainly $f_n(x) = 0$ for $x \leq 0$, for all n . For $x > 0$, there is positive integer n_o such that $x > \frac{2}{n_o}$, by the Archimedean property of the reals. Then for $n \geq n_o$ the functions f_n are 0 at x , giving the pointwise convergence to 0.

By design, the integrals of all the f_n are 1. Given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|g(x) - g(0)| < \varepsilon$ for $|x - 0| < \delta$. For n large enough so that $\frac{2}{n} < \delta$,

$$\begin{aligned} \left| \int_{\mathbb{R}} f_n(x) \cdot g(x) dx - g(0) \right| &= \left| \int_{\mathbb{R}} f_n(x) \cdot (g(x) - g(0)) dx \right| = \int_0^{2/n} f_n(x) \cdot |g(x) - g(0)| dx \\ &\leq \int_0^{2/n} f_n(x) \cdot \varepsilon dx = \varepsilon \cdot \int_0^{2/n} f_n(x) dx = \varepsilon \end{aligned}$$

giving the convergence of the integrals to $g(0)$ as claimed. ///

[01.3] Show that the functions $f_n(x) = t_{\frac{1}{n},n^2}(x - \frac{1}{n}) - t_{\frac{1}{n},n^2}(x + \frac{1}{n})$, whose graphs are tall tents of area n upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

$$\lim_n \int_{\mathbb{R}} f_n(x) \cdot g(x) dx = 2 \cdot g'(0) \quad (\text{for differentiable } g \text{ with derivative } g' \text{ in } C^o(\mathbb{R}))$$

Discussion: Certainly $f_n(x) = 0$ for $|x| \geq \frac{2}{n}$, for all n , and $f_n(0) = 0$ for all n . Given $x \neq 0$, there is positive integer n_o such that $|x| > \frac{2}{n_o}$, by the Archimedean property of the reals. Then for $n \geq n_o$ the functions f_n are 0 at x , giving the pointwise convergence to 0.

By a Taylor-Maclaurin expansion, $g(x) = g(0) + g'(0) \cdot x + h(x)$ where $|h(x)/x| \rightarrow 0$ as $x \rightarrow 0$. By linearity of integrals,

$$\begin{aligned} \int_{\mathbb{R}} f_n(x) \cdot g(x) dx &= \int_{\mathbb{R}} f_n(x) \cdot g(0) dx + \int_{\mathbb{R}} f_n(x) \cdot g'(0)x dx + \int_{\mathbb{R}} f_n(x) \cdot h(x) dx \\ &= g(0) \int_{\mathbb{R}} f_n(x) dx + g'(0) \int_{\mathbb{R}} f_n(x) \cdot x dx + \int_{\mathbb{R}} f_n(x) \cdot h(x) dx \\ &= g(0) \cdot 0 + g'(0) \int_{\mathbb{R}} f_n(x) \cdot x dx + \int_{\mathbb{R}} f_n(x) \cdot h(x) dx = g'(0) \int_{\mathbb{R}} f_n(x) \cdot x dx + \int_{\mathbb{R}} f_n(x) \cdot h(x) dx \end{aligned}$$

Much as in the previous example, for the integral involving h , given $\varepsilon > 0$, let $\delta > 0$ be small enough so that $|h(x)/x| < \varepsilon$ for $0 < |x - 0| < \delta$. For n large enough so that $\frac{2}{n} < \delta$,

$$\begin{aligned} \left| \int_{\mathbb{R}} f_n(x) \cdot h(x) dx \right| &= \int_{\mathbb{R}} |f_n(x)| \cdot |h(x)| dx = \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot |h(x)| dx \leq \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot \varepsilon |x| dx \\ &= \varepsilon \int_{|x| \leq \frac{2}{n}} |f_n(x)| \cdot |x| dx \end{aligned}$$

Thus, evaluation of $\int f_n \cdot x$ will also facilitate showing $\int f_n \cdot h \rightarrow 0$. Since both f_n and x are *odd*, their product is *even*, so

$$\int_{|x| \leq \frac{2}{n}} f_n(x) \cdot x dx = 2 \cdot \int_0^{2/n} f_n(x) \cdot x dx = 2 \cdot \int_0^{1/n} n^3 x \cdot x dx + 2 \cdot \int_{1/n}^{2/n} (n^2 - n^3(x - \frac{1}{n})) \cdot x dx$$

Replacing x by $x + \frac{1}{n}$ in the second integral gives some simplification:

$$2 \cdot \int_0^{1/n} n^3 x \cdot x dx + 2 \cdot \int_0^{1/n} (n^2 - n^3 x) \cdot (x + \frac{1}{n}) dx = 2 \cdot \int_0^{1/n} n^2 x + (n - n^2 x) dx = 2 \int_0^{1/n} n dx = 2$$

Thus, $|\int f_n \cdot h| < \varepsilon \cdot 1$ for $n \geq 2/\delta$, so that integral goes to 0, and the coefficient of $g'(0)$ is 2. ///

[01.4] Show that the closed unit ball in ℓ^2 , although *closed* and *bounded*, is *not compact*, by showing it is not *sequentially compact*.

Discussion: Let $e_n = (0, \dots, 0, 1, 0, \dots)$ with the single 1 at the n^{th} place. Then $d(e_m, e_n) = \sqrt{2}$ for $m \neq n$. Thus, the sequence of e_n 's has no Cauchy subsequence, so no convergent subsequence. ///

[01.5] Show that the *Hilbert cube*

$$C = \{(z_1, z_2, \dots) \in \ell^2 : |z_n| \leq \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals ε_n ,

$$C(\varepsilon) = \{(z_1, z_2, \dots) \in \ell^2 : |z_n| \leq \varepsilon_n\}$$

is compact if and only if $\sum_n |\varepsilon_n|^2 < \infty$.

Discussion: Probably better to rewrite $C(\varepsilon)$ as $C(\delta)$ with $\delta = (\delta_1, \delta_2, \dots)$. Use the *total boundedness* criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is n_o large enough so that $\sum_{n \geq n_o} \delta_n^2 < \varepsilon^2$. The set

$$C_{n_o} = \{(z_1, z_2, \dots, z_{n_o}) \in \mathbb{R}^{n_o} : |z_n| \leq \delta_n\}$$

is a compact subset of \mathbb{R}^{n_o} , so certainly has a finite cover by open balls of radius ε . Let the centers of these balls be w_1, \dots, w_N . Let $j : \mathbb{R}^{n_o} \rightarrow \ell^2$ be the inclusion $j(z_1, \dots, z_{n_o}) = (z_1, \dots, z_{n_o}, 0, 0, \dots)$. Then we claim that the open balls of radius 2ε at $j(w_1), j(w_2), \dots, j(w_N)$ cover $C(\delta)$. Indeed, given $z = (z_1, z_2, \dots) \in C(\delta)$, write $z = j(z') + z''$ where $z' = (z_1, \dots, z_{n_o})$ and $z'' = z - j(z') = (0, \dots, 0, z_{n_o+1}, \dots)$. There is at least one of the w_j s within ε of z' : let w_{j_o} be such. By the triangle inequality for the norm $|\cdot|_{\ell^2}$ on ℓ^2 ,

$$\begin{aligned} d(z, j(w_{j_o})) &= |z - j(w_{j_o})|_{\ell^2} = |j(z') + z'' - j(w_{j_o})|_{\ell^2} \leq |j(z') - j(w_{j_o})|_{\ell^2} + |z''|_{\ell^2} \\ &= |z' - w_{j_o}|_{\mathbb{R}^{n_o}} + |z''|_{\ell^2} < \varepsilon + \varepsilon \end{aligned}$$

Thus, $C(\delta)$ can be covered by finitely-many open balls of radius 2ε . ///

[01.6] Show that a closed interval $[a, b]$ has the expected Lebesgue outer measure, namely, $|b - a|$, by showing that the *inf* of $\sum_{j=1}^n |b_n - a_n|$ for all finite open covers $[a, b] \subset \bigcup_{j=1}^n (a_j, b_j)$ is $|b - a|$.

Discussion: There is no need to use more than one copy of a given interval in approximating the *inf*, so we can assume the intervals are distinct. Further, we can assume that the finite cover is *minimal* in the sense that no (a_i, b_i) is redundant.

Renumbering if necessary, suppose that $a_1 \leq a_i$ for all $i = 1, \dots, n$. Necessarily $a_1 < a$, or else the left endpoint of $[a, b]$ is not covered. Without loss of generality, $b_1 > a$, or else we could have dropped (a_1, b_1) from the cover. Then there is at least one index $j > 1$ such that $a_j < b_1$, or else (a_1, b_1) and $\bigcup_{j>1} (a_j, b_j)$ are disjoint, so $[a, b] \cap (a_1, b_1)$ and $[a, b] \cap \bigcup_{j>1} (a_j, b_j)$ are disjoint and non-empty, contradicting the connectedness of $[a, b]$. Renumbering if necessary, $j = 2$. Then $b_2 > b_1$, or else we could have dropped (a_2, b_2) from the cover. In fact, $a_2 > a_1$, or else (a_1, b_1) is redundant.

Suppose by induction, we have renumbered so that $a_1 < a_2 < \dots < a_m$, $b_1 < \dots < b_m$, $a_{i+1} < b_i$ for $i = 1, \dots, m-1$, and $a_1 < a$. If $b_m > b$, we are already done. For $b_m \leq b$, we claim there is an index $j > m$ such that $a_j < b_m$. If not, $\bigcup_{i \leq m} (a_i, b_i)$ and $\bigcup_{j > m} (a_j, b_j)$ are disjoint, meeting $[a, b]$ in non-empty open subsets, contradicting the connectedness of $[a, b]$. Renumber if necessary so that $j = m+1$. Necessarily $b_{m+1} > b_m$, or else (a_{m+1}, b_{m+1}) was redundant. Thus, by induction, we have $a_1 < a_2 < \dots < a_n$, $b_1 < \dots < b_n$, $a_{i+1} < b_i$ for $i = 1, \dots, m-1$, $a_1 < a$, and $b_n > b$. Then

$$\begin{aligned} \sum_{i=1}^n b_i - a_i &= \sum_{i=1}^{n-1} (b_i - a_i) + (b_n - a_n) \geq \sum_{i=1}^{n-1} (a_{i+1} - a_i) + (b_n - a_n) \\ &= (a_2 - a_1) + (a_3 - a_2) + \dots + (a_n - a_{n-1}) + (b_n - a_n) = b_n - a_1 \geq b - a \end{aligned}$$

as claimed. ///

[01.7] Let f be a continuous function on $[0, 1]$, with $f(0) = 0$ and $f(1) = 1$. Show that $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive Lebesgue measure.

Discussion: By the intermediate value theorem, that set is not empty. It is the inverse image of an open set, so by continuity is open. Thus, it contains a non-empty interval, which has positive measure. ///