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Examples discussion 02

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-02.pdf]

[02.1] Show that ℓ^2 is *complete* as a metric space.

Discussion: We can do this directly, although it is also a special case of the general fact that $L^2(X, \mu)$ is complete. Indeed, the argument will be a somewhat simpler version of the more general proof.

Let $f_1, f_2, \ldots$ be a Cauchy sequence in ℓ^2. Let $f(n)$ be the n^{th} component of $f \in \ell^2$, for $n = 1, 2, \ldots$. For any $f \in \ell^2$, certainly $|f(n)| \leq |f|_{\ell^2}$, so for each n the scalar sequence $f_1(n), f_2(n), f_3(n), \ldots$ must be Cauchy, thus has a limit $f(n)$. We claim that $f = (f(1), f(2), f(3), \ldots)$ is in ℓ^2, and is the ℓ^2 limit of the f_i.

Given $\varepsilon > 0$, there is N sufficiently large so that $|f_i - f_j|_{\ell^2} < \varepsilon$ for all $i, j \geq N$. By a discrete version of Fatou's lemma, for $i \geq N$,

$$\sum_n |f(n) - f_i(n)|^2 \;=\; \sum_n \lim_j |f_j(n) - f_i(n)|^2 \;=\; \sum_n \liminf_j |f_j(n) - f_i(n)|^2 \;\leq\; \liminf_j \sum_n |f_j(n) - f_i(n)|^2$$

$$\leq\; \liminf_j |f_j - f_i|^2_{\ell^2} \;\leq\; \liminf_j \varepsilon^2 \;=\; \varepsilon^2$$

Thus, $f - f_i \in \ell^2$, so $f = (f - f_i) + f_i \in \ell^2$. Then the previous computation shows that for given ε for $i \geq N$ we have $|f - f_i| \leq \varepsilon$. Thus, $f_i \to f$ in ℓ^2. ///

Discrete version of Fatou's Lemma: We claim that for $[0, +\infty]$-valued functions f_j on $\{1, 2, 3, \ldots\}$,

$$\sum_{n=1}^{\infty} \liminf_j f_j(n) \;\leq\; \liminf_j \sum_{n=1}^{\infty} f_j(n)$$

Proof: Letting $g_j(n) = \inf_{i \geq j} f_j(n)$, certainly $g_j(n) \leq f_j(n)$ for all n, and $\sum_n g_j(n) \leq \sum_n f_j(n)$. Also, $g_1(n) \leq g_2(n) \leq \ldots$ for all n, and $\lim_j g_j(n) = \liminf_j f_j(n)$. A discrete form of the Monotone Convergence Theorem, proven just below, is

$$\sum_n \lim_j g_j(n) \;=\; \lim_j \sum_n g_j(n)$$

Thus,

$$\sum_n \liminf_j f_j(n) \;=\; \sum_n \lim_j g_j(n) \;=\; \lim_j \sum_n g_j(n) \;=\; \liminf_j \sum_n g_j(n) \;\leq\; \liminf_j \sum_n f_j(n)$$

as claimed. ///

Similarly, we have

Discrete version of Lebesgue's Monotone Convergence Theorem: For $[0, +\infty]$-valued functions f_j on $\{1, 2, 3, \ldots\}$, with $f_1(n) \leq f_2(n) \leq \ldots$ for all n,

$$\lim_j \sum_{n=1}^{\infty} f_j(n) \;=\; \sum_{n=1}^{\infty} \lim_j f_j(n) \qquad \text{(allowing value } +\infty)$$

Proof: Each non-decreasing sequence $f_1(n) \leq f_2(n) \leq \ldots$ has a limit $f(n) \in [0, +\infty]$. Similarly, since $\sum_n f_j(n) \leq \sum_n f_{j+1}(n)$, the non-decreasing sequence of these sums has a limit $S = \lim_j \sum_n f_j(n)$. Since $f_j(n) \leq f(n)$, certainly $\sum_n f_j(n) \leq \sum_n f(n)$, and $S \leq \sum_n f(n)$.

Fix N, and put $g(n) = f(n)$ for $n \le N$ and $g(n) = 0$ for $n > N$. For $\varepsilon > 0$, let

$$E_j \;=\; \{n : \sum_n f_j(n) \ge (1-\varepsilon)\cdot \sum_n g(n)\} \qquad\qquad (\text{for } j = 1, 2, \ldots)$$

Certainly $E_1 \subset E_2 \subset \ldots$, since $f_{j+1}(n) \ge f_j(n)$ for all n. We claim that $\bigcup E_j = \{1, 2, \ldots\}$: for $f(n) > 0$,

$$\lim_j f_j(n) \;=\; f(n) \;>\; (1-\varepsilon)\cdot f(n) \;\ge\; (1-\varepsilon)\cdot g(n) \qquad\qquad (\text{for all } n)$$

and for $f(n) = 0$, also $g(n) = 0$, and

$$f_1(n) \;\ge\; 0 \;\ge\; (1-\varepsilon)\cdot g(n)$$

Then

$$\sum_n f_j(n) \;\ge\; \sum_{n\in E_j} f_j(n) \;\ge\; (1-\varepsilon)\cdot \sum_{n\in E_j} g(n)$$

The set of n for which $g(n)$ is non-zero is finite, so there is j_o such that for $j \ge j_o$

$$\sum_{n\in E_j} g(n) \;=\; \sum_n g(n) \qquad\qquad (\text{for all } j \ge j_o)$$

That is, $\lim_j \sum_n f_j(n) \ge (1-\varepsilon)\sum_n g(n)$. Then

$$S \;=\; \lim_j \sum_n f_j(n) \;\ge\; (1-\varepsilon)\cdot \lim_j \sum_{n\in E_j} g(n) \;=\; (1-\varepsilon)\cdot \sum_n g(n)$$

This holds for every $\varepsilon > 0$, so $S \ge \sum_n g(n) = \sum_{n\le N} f(n)$. This holds for every N, so $S \ge \sum_n f(n)$. ///

[02.2] Show that the characteristic function χ_E of a measurable set E is measurable.

Discussion: For non-empty open $U \subset \mathbb{R}$, $\chi_E^{-1}(U)$ is the measurable set ϕ if U does not contain either 0 or 1. If $U \ni 1$ but $U \not\ni 0$, then $\chi_E^{-1}(U) = E$, which is measurable. If $U \ni 0$ but $U \not\ni 1$, then $\chi_E^{-1}(U) = E^c$, the complement of E, which is measurable. If U contains both 0 and 1, then $\chi_E^{-1}(U)$ is the whole domain space, which is measurable. ///

[02.3] Show that the product of two $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ is measurable.

Discussion: Let f, g be measurable functions. Let $\Delta : \mathbb{R} \to \mathbb{R}^2$ by $\Delta(x) = (x, x)$, $s : \mathbb{R}^2 \to \mathbb{R}$ by $m(x, y) = x \cdot y$, and $f \oplus g : \mathbb{R}^2 \to \mathbb{R}^2$ by $(f \oplus g)(x, y) = (f(x), g(y))$. Clearly $m \circ (f \oplus g) \circ \Delta = f \cdot g$, and $(f \cdot g)^{-1} = \Delta^{-1} \circ (f \oplus g)^{-1} \circ m^{-1}$.

For open $U \subset \mathbb{R}$, $m^{-1}(U) \subset \mathbb{R}^2$ is open, because m is continuous. Since $\mathbb{R}^2$ is countably based, and in fact has a countable basis consisting of rectangles with rational endpoints, so $m^{-1}(U)$ is a countable unions of rectangles $(a_i, b_i) \times (c_i, d_i)$. Then

$$(f \oplus g)^{-1} \circ m^{-1}(U) \;=\; (f \oplus g)^{-1}(\bigcup_i (a_i, b_i) \times (c_i, d_i))$$

$$=\; \bigcup_i (f \oplus g)^{-1}((a_i, b_i) \times (c_i, d_i)) \;=\; \bigcup_i f^{-1}(a_i, b_i) \times g^{-1}(c_i, d_i)$$

The sets $f^{-1}(a_i, b_i) \subset \mathbb{R}$ and $g^{-1}(c_i, d_i) \subset \mathbb{R}$ are Borel sets, so their product is a Borel set in $\mathbb{R}^2$. Then

$$\Delta^{-1}(E_1 \times E_2) \;=\; E_1 \cap E_2 \qquad\qquad (\text{for } E_1, E_2 \text{ measurable in } \mathbb{R})$$

is measurable. ///

[02.4] Use Urysohn's lemma to prove that $C^o[a,b]$ is dense in $L^1[a,b]$.

Discussion: By the Lebesgue definition of integrals, *simple* functions are dense in $L^1[a,b]$, so it suffices to show that *simple* functions can be well approximated by continuous functions. Granting ourselves the *(outer and inner) regularity* of Lebesgue measure μ, for measurable E there are open U and compact K such that $K \subset E \subset U$, and $\mathfrak{m}(U) - \mu(K) < \varepsilon$. Invoke Urysohn to make a continuous function f taking values in $[0,1]$ and $f|_K = 1$ and $f = 0$ off U. Then

$$\int_a^b |f - \mathrm{ch}_E| \;=\; \int_K |f - \mathrm{ch}_E| + \int_{E-K} |f - \mathrm{ch}_E| + \int_{U-E} |f - \mathrm{ch}_E| \;\le\; \int_K |1-1| + \int_{E-K} 1 + \int_{U-E} 1$$
$$= \;\mu(E-K) + \mu(U-E) \;=\; \mu(U-K) \;<\; \varepsilon$$

as desired. ///

[02.5] Comparing L^p spaces. Let $1 \le p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

Discussion: Take $p < p'$. We claim that $L^p[a,b] \supset L^{p'}[a,b]$, with proper containment. The function f that is $(x-a)^{-\frac{1}{p'}}$ on $(a,b]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p. Given $f \in L^{p'}[a,b]$, let E be the set of $x \in [a,b]$ where $|f(x)| \ge 1$. Then $\int_a^b |f|^{p'} < \infty$ if and only if $\int_E |f|^{p'} < \infty$. On E, $|f|^p < |f|^{p'}$, so $\int_E |f|^p < \infty$, and then also $\int_a^b |f|^p < \infty$, so $f \in L^p[a,b]$. ///

We claim that $L^p(\mathbb{R})$ and $L^{p'}(\mathbb{R})$ are not comparable for $p \ne p'$. Take $1 \le p < p'$. On one hand, $1/(1+|x|)^{1/p'+\varepsilon}$ is in $L^{p'}$ for all $\varepsilon > 0$, but not in L^p for ε small enough so that $\frac{1}{p'} + \varepsilon < \frac{1}{p}$. On the other hand, the function f that is $x^{-\frac{1}{p'}}$ on $(0,1]$ and 0 off that interval is *not* in $L^{p'}$, but is in L^p.

We claim that for $1 \le p < p' < \infty$, $\ell^p \subset \ell^{p'}$, with strict containment. Indeed, $f(n) = 1/n^p$ is not in ℓ^p, but is in $\ell^{p'}$. Let $E = \{n \in \{1,2,\ldots\} : |f(n)| < 1\}$. Then $f \in \ell^p$ if and only if the *complement* of E is finite, and if $\sum_{n \in E} |f(n)|^p < \infty$. Certainly $|f(n)|^p > |f(n)|^{p'}$ for $n \in E$, and the complement of E is finite, so $\sum_{n\in E} |f(n)|^{p'} < \sum_{n \in E} |f(n)|^p$, and $f \in \ell^{p'}$. ///

[02.6] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\le\; w_1 a_1 + \ldots + w_n a_n$$

Discussion: This is a corollary of Jensen's inequality, similar to the arithmetic-geometric mean, but with unequal weights. Namely, let $X = \{1, 2, \ldots, n\}$ with measure $\mu(i) = w_i$, and function $f(i) = \log a_i$. Then Jensen's inequality is

$$\Big(\sum_{i=1}^n w_i \cdot \log a_i\Big) \;=\; \sum_{i=1}^n w_i \cdot e^{\log a_i}$$

which simplifies to the assertion. ///

[02.7] In ℓ^2, show that the point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

Discussion: The minimum principle assures that there is a *unique* closest point w in the closed unit ball B to v, because B is convex, closed, non-empty, and v is not in B.

Suppose w is closer than $v/v|$. Then

$$|v|^2 - 2|v| + 1 \;=\; |v - \frac{v}{|v|}|^2 \;>\; |v-w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + |w|^2 \;=\; |v|^2 - \langle v,w\rangle - \langle w,v\rangle + 1$$

Thus,

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle$$

Thus, the sum of the two inner products is *positive*, and by Cauchy-Schwarz-Bunyakowsky:

$$2|v| \;<\; \langle v,w\rangle + \langle w,v\rangle \;=\; |\langle v,w\rangle + \langle w,v\rangle| \;\le\; 2|v|\cdot|w|$$

Thus, $1 < |w|$, which is impossible. ///

[02.8] For a measurable set $E \subset [0,2\pi]$, show that

$$\lim_{n\to\infty} \int_E \cos nx\; dx \;=\; 0 \;=\; \lim_{n\to\infty} \int_E \sin nx\; dx$$

Discussion: This is an instance of a *Riemann-Lebesgue lemma*, namely, that Fourier coefficients of an L^2 function on $[0,2\pi]$ go to 0. Here, the L^2 function is the characteristic function of E, and we use sines and cosines instead of exponentials. ///

[02.9] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0,2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. Write out the conclusion of Parseval's theorem for this function.

Discussion: We have the orthonormal basis $e_n(x) = \frac{1}{\sqrt{2\pi}} e^{inx}$ with $n \in \mathbb{Z}$ for the Hilbert space $L^2[0,2\pi]$. The Fourier coefficients are determined by Fourier's formula

$$\widehat{f}(n) \;=\; \int_0^{2\pi} f(x)\, \frac{e^{-inx}}{\sqrt{2\pi}}\; dx$$

For $n = 0$, this is 0. For $n \neq 0$, integrate by parts, to get

$$\widehat{f}(n) \;=\; \Big[f(x)\cdot \frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\Big]_0^{2\pi} - \int_0^{2\pi} 1\cdot\, \frac{e^{-inx}}{\sqrt{2\pi}\cdot(-in)}\; dx$$

$$= \Big((\pi\cdot \frac{1}{\sqrt{2\pi}\cdot(-in)}) - (-\pi\cdot\frac{1}{\sqrt{2\pi}\cdot(-in)})\Big) - 0 \;=\; \frac{2\pi}{\sqrt{2\pi}\cdot(-in)} \;=\; \frac{\sqrt{2\pi}}{-in}$$

The L^2 norm of f is

$$\int_0^{2\pi} (x-\pi)^2\; dx \;=\; \Big[\frac{(x-\pi)^3}{3}\Big]_0^{2\pi} \;=\; \frac{\pi^3 - (-\pi)^3}{3} \;=\; \frac{2\pi^3}{3}$$

Thus, by Parseval,

$$\sum_{n\neq 0} \Big|\frac{\sqrt{2\pi}}{-in}\Big|^2 \;=\; \frac{2\pi^3}{3}$$

This simplifies first to

$$2\sum_{n\ge 1} \frac{2\pi}{n^2} \;=\; \frac{2\pi^3}{3}$$

and then to

$$\sum_{n\ge 1} \frac{1}{n^2} \;=\; \frac{\pi^2}{6}$$

That is, Parseval applied to the sawtooth function evaluates $\zeta(2)$. ///

[02.10] For fixed $y \in [0,1]$, show that there is *no* $f_y \in L^2[0,1]$ so that $\langle g, f_y\rangle = g(y)$ for all $g \in L^2[0,1]$.

Discussion: Part of the issue here is whether L^2 functions truly have meaningful pointwise values at all, and we generally imagine that they do *not*, although such a negative fact may be hard to express formulaically.

Among many approaches, one is to suppose such f exists. Choose an orthonormal basis for $L^2[0,1]$ consisting of the continuous functions $\psi_n(x) = e^{2\pi i n x}$, and see what the condition $\langle f_y, \psi_n\rangle = \psi_n(y)$ imposes on the alleged f_y. Indeed, this condition completely determines the Fourier coefficients of the alleged f_y: since $\psi_n \in L^2[0,1]$, $\langle \psi_n, f_y\rangle = \psi_n(y)$, and then

$$\overline{\widehat{f}_y(n)} \;=\; \int_0^1 \overline{f_y(x)\,\overline{\psi}_n(x)}\,dx \;=\; \langle \psi_n, f_y\rangle \;=\; \psi_n(y)$$

so

$$f_y \;=\; \sum_{n\in\mathbb{Z}} \overline{\psi}_n(y)\cdot\psi_n \qquad\qquad \text{(with equality in an } L^2 \text{ sense)}$$

By Parseval,

$$|f_y|^2_{L^2} \;=\; \sum_n |\psi_n(y)|^2 \;=\; +\infty$$

since $|\psi_n(y)| = 1$ for all n. Thus, there can be no such f_y in L^2. ///

In contrast to the previous example's outcome: Let V be the complex vector space of power series $f(z) = \sum_{n\ge 0} c_n\, z^n$ convergent on the open unit disk D in $\mathbb{C}$, having finite norm

$$|f| \;=\; \Big(\int_D |f(x+iy)|^2\;dx\,dy\Big)^{\frac{1}{2}}$$

with hermitian inner product

$$\langle f, g\rangle \;=\; \int_D f(x+iy)\cdot\overline{g(x+iy)}\;dx\,dy$$

It is not hard to show that $\langle z^m, z^n\rangle = 0$ unless $m = n$, in which case it is $\frac{2\pi}{2n+1}$, and that $\psi_n(z) = z^n\cdot\frac{\sqrt{2n+1}}{\sqrt{2\pi}}$ is an orthonormal basis for V. The sum $f_w(z) = \sum_{n\ge 0}\psi_n(z)\,\overline{\psi_n(w)}$ converges absolutely for $z, w \in D$, and

$$\langle g(-),\, f_w\rangle \;=\; g(w) \qquad\qquad \text{(for } w \text{ in the disk)}$$

For each fixed $w \in D$, pointwise evaluation $g \to g(w)$ is a continuous linear functional on V.