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Examples discussion 03

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http://www.math.umn.edu/˜garrett/m/real/examples_2017-18/real-disc-03.pdf]

[03.1] Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) *closed*.

Discussion: Let $v_1, \ldots, v_m$ be an orthonormal basis for the given vector subspace W. For a Cauchy sequence $\{w_n\}$ in W, we claim that for each j the sequence $\langle w_n, v_j\rangle$ is Cauchy: by Cauchy-Schwarz-Bunyakowsky,

$$|\langle w_n, v_j\rangle - \langle w_{n'}, v_j\rangle| \;=\; |\langle w_n - w_{n'}, v_j\rangle| \;\le\; |w_n - w_{n'}| \cdot |v_j| \;=\; |w_n - w_{n'}|$$

Thus, by completeness of $\mathbb{R}$ and/or $\mathbb{C}$, that sequence has a limit c_j. As expected, we claim that $\lim_n w_n = \sum_{j=1}^m c_j \cdot v_j$. Indeed, using the orthonormality of the v_j's,

$$\Big|w_n - \sum_{j=1}^m c_j \cdot v_j\Big|^2 \;=\; \Big|w_n - \sum_{j=1}^m \lim_i \langle w_i, v_j\rangle \cdot v_j\Big|^2 \;=\; \Big|\sum_{j=1}^m \lim_i \langle w_n - w_i, v_j\rangle \cdot v_j\Big|^2$$

$$\le\; \sum_{j=1}^m |\lim_i \langle w_n - w_i, v_j\rangle|^2 \;=\; \lim_i \sum_{j=1}^m |\langle w_n - w_i, v_j\rangle|^2 \;\le\; \lim_i \sum_{j=1}^m |w_n - w_i| \cdot |v_j| \;=\; \lim_i m \cdot |w_n - w_i|$$

Take n_o large enough so that $|w_n - w_i| < \varepsilon$ for $i, n \ge n_o$. Then the latter expression is at most $m \cdot \varepsilon$. This holds for all $\varepsilon > 0$, so the limit is 0. ///

[03.2] For a subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the closure of the subspace W in V.

Discussion: Let $\lambda_x(v) = \langle v, x\rangle$ for $x, v \in V$. Then $W^\perp = \bigcap_{w\in W} \ker \lambda_w$. Similarly, $(W^\perp)^\perp = \bigcap_{x \in W^\perp} \ker \lambda_x$. From the discussion in the Riesz-Fréchet theorem, or directly via Cauchy-Schwarz-Bunyakowsky, each λ_x is continuous, so $\ker \lambda_x = \lambda_x^{-1}(\{0\})$ is closed, since $\{0\}$ is closed. (One might check that the kernel of a linear map is a vector subspace.) An arbitrary intersection of closed sets is closed, so $(W^\perp)^\perp$ is closed.

Certainly $(W^\perp)^\perp \supset W$, because for each $w \in W$, $\langle x, w\rangle = 0$ for all $x \in W^\perp$. Thus, $(W^\perp)^\perp$ is a closed subspace, containing W. Being a closed subspace of a Hilbert space, $(W^\perp)^\perp$ is a Hilbert space itself. If $(W^\perp)^\perp$ were strictly larger than the topological closure $\overline{W}$ of W, then there would be $0 \ne y \in (W^\perp)^\perp$ orthogonal to $\overline{W}$. Then y would be orthogonal to W itself, so $0 \ne y \in W^\perp$, contradicting $0 \ne y \in (W^\perp)^\perp$. ///

[03.3] Show that for $0 < x < 1$

$$\sum_{n\ge 1} \frac{\sin 2\pi n x}{n} \;=\; \pi(\tfrac{1}{2} - x)$$

Discussion: The Fourier series of the right-hand side is computed to be that given on the left-hand side. By the Fourier-Dirichlet result on pointwise convergence, since $\pi(\frac{1}{2} - x)$ is finitely-piecewise C^o, and has left and right derivatives in $(0, 1)$, its Fourier series converges to it pointwise there. ///

[03.4] Let $c_1, c_2, \ldots$ be positive real, converging monotonically to 0. For $0 < x < 1$, prove that $\sum_{n\ge 0} c_n\, e^{2\pi i n x}$ *converges pointwise*.

Discussion: The expression as a Fourier series should not distract us from seeing an instance of the generalized alternating-decreasing criterion again, sometimes called *Dirichlet's criterion*: for a positive real sequence $c_1, c_2, \ldots$ monotone-decreasing to 0, and for a (possibly complex) sequence $b_1, b_2, \ldots$ with *bounded* *partial sums* $B_n = b_1 + \ldots + b_n$, the sum $\sum_n b_n c_n$ converges. The partial sums $\sum_{n\le N} e^{2\pi i n x}$ are bounded for $0 < x < 1$, by summing geometric series, so this criterion applies here.

The proof of the criterion itself is by *summation by parts*, a discrete analogue of integration by parts. That is, rewrite the tails of the sum as

$$\sum_{M\le n\le N} b_n c_n \;=\; \sum_{M\le n\le N} (B_n - B_{n-1})c_n \;=\; -B_{M-1}c_M + \sum_{M\le n\le N} B_n(c_n - c_{n+1}) + B_N c_{N+1}$$

Since the partial sums are bounded, the first and last summand go to 0. Letting β be a bound for all the $|B_n|$, the summation is

$$\Big|\sum_{M\le n\le N} B_n(c_n - c_{n+1})\Big| \;\le\; \sum_{M\le n\le N} |B_n|\cdot|c_n - c_{n+1}| \;=\; \sum_{M\le n\le N} |B_n|\cdot(c_n - c_{n+1}) \;\le\; \sum_{M\le n\le N} \beta\cdot(c_n - c_{n+1})$$

$$=\; \beta\cdot\sum_{M\le n\le N} (c_n - c_{n+1}) \;=\; \beta\cdot(c_M - c_{N+1})$$

by telescoping the series. Again, c_M and c_{N+1} go to 0. ///

[03.5] Show that the sup-norm completion of the space $C^o_c(\mathbb{R})$ of compactly-supported continuous functions is the space $C^o_o(\mathbb{R})$ of continuous functions going to 0 at infinity. An analogous assertion and argument should hold for any topological space in place of $\mathbb{R}$.

Discussion: The argument for this is general enough that we can replace $\mathbb{R}$ by a more general topological space X, probably locally compact and Hausdorff so that Urysohn's lemma assures us a good supply of continuous functions for auxiliary purposes. Then $C^o_o(X)$ is defined to be the collection of continuous functions f such that, given $\varepsilon > 0$, there is a compact $K \subset X$ such that $|f(x)| < \varepsilon$ for $x \notin K$.

First, show that any $f \in C^o_o(\mathbb{R})$ is a sup-norm limit of functions from $C^o_c(\mathbb{R})$. Given $\varepsilon > 0$, let K be sufficiently large so that $|f(x)| < \varepsilon$ for $x \notin K$. We claim that there is an open $U \supset K$ with compact closure $\overline{U}$ (which would be obvious on $\mathbb{R}$ or $\mathbb{R}^n$). For each $x \in K$, let $U_x \ni x$ be an open set with compact closure (using the local compactness). By compactness of K, there is a finite subcover $K \subset U_{x_1} \cup \ldots \cup U_{x_n}$. Then the closure of $U = U_{x_1} \cup \ldots \cup U_{x_n}$ is compact, as claimed. Then, invoking Urysohn's Lemma, let φ be a continuous function on X taking values in the interval $[0, 1]$, that is 1 on K, and 0 off U, so φ has compact support. Then $\varphi \cdot f$ is continuous and has compact support, and

$$\sup_{x\in X} |f(x) - \varphi(x)\cdot f(x)| \;\le\; \sup_{x\in K} |f(x) - \varphi(x)\cdot f(x)| + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)| \;=\; 0 + \sup_{x\notin K} |f(x) - \varphi(x)\cdot f(x)|$$

$$\le\; \sup|1-\varphi|\cdot\sup_{x\notin K}|f(x)| \;<\; 1\cdot\varepsilon$$

That is, we can approximate f to within ε, as claimed.

On the other hand, now show that any sup-norm Cauchy sequence of $f_n \in C^o_c(X)$ has a pointwise limit f in $C^o_o(X)$. First, on any compact, the limit of the f_n's is *uniform* pointwise, so is continuous on compacts. Since every point $x \in X$ has a neighborhood U_x with compact closure, the pointwise limit is continuous on U_x. Thus, the pointwise limit is continuous at every point, hence continuous. Given $\varepsilon > 0$, take n_o sufficiently large so that $\sup_{x\in X} |f_m(x) - f_n(x)| < \varepsilon$ for all $m, n \ge n_o$. Let K be the support of f_{n_o}. Then

$$\sup_{x\notin K} |f(x)| \;=\; \sup_{x\notin K} |f(x) - f_{n_o}(x)| \;\le\; \sup_{x\in X} |f(x) - f_{n_o}| \;\le\; \varepsilon$$

Thus, the pointwise limit goes to 0 at infinity. ///

[03.6] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2\, dx$. (*Hint:* use Plancherel.)

Discussion: From a standard stock of easy Fourier transforms, the Fourier transform of a characteristic function of a symmetrical interval is very close to the given function:

$$\widehat{\mathrm{ch}_{[-1,1]}}(\xi) \;=\; \int_{-1}^{1} e^{-2\pi i \xi x}\,dx \;=\; \frac{e^{-2\pi i\xi} - e^{2\pi i \xi}}{-2\pi i \xi} \;=\; \frac{\sin 2\pi\xi}{\pi\xi}$$

Applying Plancherel, we have

$$2 \;=\; \int_{\mathbb{R}} |\mathrm{ch}_{[-1,1]}|^2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin 2\pi\xi}{\pi\xi}\Big)^2\,d\xi$$

The change of variables replacing ξ by $\xi/2\pi$ gives

$$2 \;=\; \int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi/2}\Big)^2 \frac{d\xi}{2\pi} \;=\; \frac{2}{\pi}\int_{\mathbb{R}} \Big(\frac{\sin\xi}{\xi}\Big)^2\,d\xi$$

Thus, the desired integral is π. ///

[03.7] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\,dx\Big| \;<\; C\cdot\sqrt{\delta}$$

Formulate and prove the corresponding assertion for L^p with $1 < p < \infty$.

Discussion: Let h_δ be the characteristic function of $[t-\delta, t+\delta]$. By Cauchy-Schwarz-Bunyakowsky

$$\Big|\int_{t-\delta}^{t+\delta} f\Big| \;=\; |\langle f, h_\delta\rangle_{L^2}| \;\le\; |f|_{L^2}\cdot|h_\delta|_{L^2} \;=\; |f|_{L^2}\cdot\sqrt{2\delta}$$

The case of conjugate exponents $\frac{1}{p}+\frac{1}{q} = 1$ is the same, using Hölder's inequality rather than Cauchy-Schwarz-Bunyakowsky. There is no immediate analogue for L^1, although a weaker result is possible, as in the next example. ///

[03.8] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\,dx\Big| \;<\; \varepsilon$$

Sharpen the first example to show that

$$\int_{t-\delta}^{t+\delta} f(x)\,dx \;=\; o(\sqrt{\delta}) \qquad\qquad (\text{as } \delta\to 0^+)$$

where Landau's little-o notation is that $f(x) = o(g(x))$ as $x \to a$ when $\lim_{x\to a} f(x)/g(x) = 0$.

Discussion: Let $S_n = \{x : \frac{1}{n+1} \le |x-t| < \frac{1}{n}\}$. Then

$$\Big|\sum_{n\ge 1}\int_{S_n} f\Big| \;\le\; \sum_{n\ge 1}\int_{S_n}|f| \;\le\; |f|_{L^1}$$

Thus, the sum of non-negative terms $\sum_{n\ge1}\int_{S_n}|f|$ is convergent, so the tails $\sum_{n\ge N}\int_{S_n}|f|$ go to 0 as $N\to+\infty$. Thus,

$$\Big|\int_{|x-t|\le 1/N} f\Big| \;\le\; \int_{|x-t|\le 1/N}|f| \;=\; \sum_{n\ge N}\int_{S_n}|f|$$

goes to 0 as $N\to+\infty$. Then this idea can be applied to $\int_{|x-t|<\delta}|f|^p$ in the previous example. ///
