(*November 2, 2017*)

Examples discussion 04

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-04.pdf]

[04.1] Fix $x_o \in [a,b]$. Show that $\lambda(f) = f(x_o)$ is a continuous linear functional on $C^o[a,b]$.

Discussion: Recall that, for linear maps, continuity is equivalent to continuity at 0, which is equivalent to being *bounded*, in the sense that there exists a constant C such that $|\lambda(f)| \le C \cdot |f|_{C^o}$ for all f. Here,

$$|\lambda(f)| \;=\; |f(x_o)| \;\le\; \sup_{x\in[a,b]} |f(x)| \;=\; |f|_{C^o}$$

so the constant $C = 1$ succeeds. ///

[04.2] Prove that Cesaro summation

$$b_1 = \frac{a_1}{1},\;\; b_2 = \frac{a_1+a_2}{2},\;\; b_3 = \frac{a_1+a_2+a_3}{3}, \ldots$$

converts every convergent sequence $a_1, a_2, \ldots$ to a convergent sequence $b_1, b_2, \ldots$ with the same limit.

Discussion: Let $\{a_n\}$ converge to A. Thus, given $\varepsilon > 0$, there is n_o be such for $n > n_o$ we have $|a_n - A| < \varepsilon$. Let $M = \max_{n\le n_o} |a_n|$. For $n \ge n_o$, by the triangle inequality,

$$|b_n - A| \;=\; \frac{|(a_1-A)+\ldots+(a_n-A)|}{n} \;\le\; \frac{|a_1|+\ldots+|a_{n_o}|}{n} + \frac{n_o\cdot|A|}{n} \;+\; \frac{|a_{n_o+1}-A|+\ldots+|a_n-A|}{n}$$

$$< \;\frac{n_o\cdot M}{n} + \frac{n_o\cdot|A|}{n} + \varepsilon$$

For n sufficiently large, depending on A, n_o and M, the sum of the first two terms can be made smaller than ε. Replace ε by $\varepsilon/2$ throughout, if desired. ///

[04.3] (Collecting Fourier transform pairs...) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x) = \begin{cases} e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x \le 0) \end{cases}$$

Discussion: The first of these is direct:

$$\widehat{\chi_{[a,b]}}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\chi_{[a,b]}(x)\,dx \;=\; \int_a^b e^{-2\pi i\xi x}\,dx \;=\; \begin{cases} \dfrac{e^{-2\pi i\xi b} - e^{2\pi i\xi a}}{-2\pi i\xi} & (\text{for } \xi \ne 0) \\ b-a & (\text{for } \xi = 0) \end{cases}$$

Since the latter function is *not* in $L^1(\mathbb{R})$, but *is* in $L^2(\mathbb{R})$, we define its Fourier transform (or inverse Fourier transform) *indirectly*, via either the inversion theorem, or by extending-by-continuity via Plancherel, expressing the function as an L^2 limite of L^1 functions.

The third is similarly direct:

$$\widehat{f}(\xi) \;=\; \int_0^\infty e^{-2\pi i\xi x}\,e^{-x}\,dx \;=\; \int_0^\infty e^{-(2\pi i\xi+1)x}\,dx \;=\; \Big[\frac{e^{-(2\pi i\xi+1)x}}{-(2\pi i\xi+1)}\Big]_0^\infty \;=\; \frac{1}{2\pi i\xi+1}$$

Again, the latter function is not in L^1, but is in L^2, so its Fourier transform is most conveniently defined indirectly.

The Gaussian's Fourier transform is less trivial to evaluate, but is a very important example to have in hand, with many different applications throughout mathematics. One approach is as follows. Letting $f(x) = e^{-\pi x^2}$,

$$\widehat{f}(\xi) \;=\; \int_{\mathbb{R}} e^{-2\pi i\xi x}\, e^{-\pi x^2}\; dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+2i\xi x)}\; dx \;=\; \int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2-\pi\xi^2}\; dx \;=\; e^{-\pi\xi^2}\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\; dx$$

by completing the square. The unobvious claim is that the integral does not depend on ξ, and, in fact, has value 1. Perhaps the optimal approach here is to observe that the integral is equal to a complex contour integral:

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\; dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\; dz$$

along the line $\mathrm{Im}(z) = i\xi$. Given the good decay of the integrand as $|\mathrm{Re}(z)| \to \infty$, by Cauchy-Goursat theory, the contour can be *moved* to integration along the real line, giving

$$\int_{\mathbb{R}} e^{-\pi(x^2+i\xi)^2}\; dx \;=\; \int_{i\xi-\infty}^{i\xi+\infty} e^{-\pi z^2}\; dz \;=\; \int_{-\infty}^{\infty} e^{-\pi x^2}\; dx \;=\; 1$$

The fact that the latter integral has value 1 comes from the usual trick involving polar coordinates:

$$\Big(\int_{-\infty}^{\infty} e^{-\pi x^2}\; dx\Big)^2 \;=\; \int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\; dx\, dy \;=\; \int_0^{2\pi}\int_0^{\infty} e^{-\pi r^2}\; r\, dr\, d\theta \;=\; 2\pi\int_0^{\infty} e^{-\pi r^2}\; r\, dr$$

Replacing r by $\sqrt{t}$, this is

$$\pi\int_0^{\infty} e^{\pi t}\; dt \;=\; \pi\cdot\frac{1}{\pi} \;=\; 1$$

Thus, with the present normalization of Fourier transform and corresponding normalization of Gaussian, the Gaussian is its own Fourier transform. ///

[04.4] Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-linear function, and express it explicitly.

Discussion: Once enunciated, this fact (and the explicit expression) should be just a matter of book-keeping. We do assume that $a \le b$ and $c \le d$. Also, by symmetry, without loss of generality we can suppose that $|b-a| \ge |d-c|$. This is used in the treatment of cases below.

$$(\chi_{[a,b]} * \chi_{[c,d]})(x) \;=\; \int_{\mathbb{R}} \chi_{[a,b]}(x-y)\cdot\chi_{[c,d]}(y)\; dy \;=\; \int_c^d \chi_{[a,b]}(x-y)\; dy$$

$$= \int_c^d \chi_{[a-x,b-x]}(-y)\; dy \;=\; \int_{-d}^{-c} \chi_{[x-b,x-a]}(y)\; dy \;=\; \mathrm{meas}\Big([-d,-c]\cap[x-b,x-a]\Big)$$

Looking at the cases of overlap, using $b-a \ge d-c$, this is

$$\begin{cases} 0 & (\text{for } x-a \le -d, \text{ that is, } [x-b,x-a] \text{ is to the left of } [-d,-c]) \\ (x-a)-(-d) & (\text{for } x-b \le -d \le x-a \le -c) \\ (-c)-(-d) & (\text{for } x-b \le -d \le -c \le x-a, \text{ that is, } [-d,-c] \subset [x-b,x-a]) \\ (-c)-(x-b) & (\text{for } -d \le x-b \le -c \le x-a) \\ 0 & (\text{for } x-b \ge -c, \text{ that is, } [x-b,x-a] \text{ is to the right of } [-d,-c]) \end{cases}$$

$$= \begin{cases} 0 & (\text{for } x \le a-d) \\ x-a+d & (\text{for } a-d \le x \le a-c) \\ d-c & (\text{for } a-c \le x \le b-d) \\ b-c-x & (\text{for } b-d \le x \le b-c) \\ 0 & (\text{for } x \ge b-c) \end{cases}$$

We used the fact that $b-a \ge d-c$ implies $a-c \le b-d$. It is useful to consider the special configuration $[a,b] = [-A,A]$ and $[c,d] = [-B,B]$ with $A \ge B \ge 0$: the convolution is

$$\begin{cases} 0 & (\text{for } x \le -A-B) \\ x+A+B & (\text{for } -A-B \le x \le -A+B) \\ 2B & (\text{for } -A+B \le x \le A-B) \\ A+B-x & (\text{for } A-B \le x \le A+B) \\ 0 & (\text{for } x \ge A+B) \end{cases}$$

In particular, the convolution is supported inside $[-A-B, A+B]$. Similarly, for f and g supported in $[-a,a]$ and $[-b,b]$, the convolution is supported in $[-a-b,\ a+b]$. ///

[04.5] Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx$$

Discussion: View this as an inner product and invoke Plancherel:

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\, dx \;=\; \Big\langle \frac{\sin x}{x},\, \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5} \Big\rangle \;=\; \Big\langle \Big(\frac{\sin x}{x}\Big)\widehat{\ },\, \Big(\frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\Big)\widehat{\ } \Big\rangle$$

Since Fourier transform converts pointwise multiplication to convolution, this is

$$\Big\langle \Big(\frac{\sin x}{x}\Big)\widehat{\ },\, \Big(\frac{\sin x/3}{x/3}\Big)\widehat{\ } * \Big(\frac{\sin x/5}{x/5}\Big)\widehat{\ } \Big\rangle$$

We have computed that

$$\widehat{\chi_{[-a,a]}}(\xi) \;=\; \frac{\sin 2\pi a\xi}{\pi\xi} \;=\; 2a \cdot \frac{\sin 2\pi a\xi}{2\pi a\xi}$$

That is, by linearity of Fourier transform,

$$\Big(\frac{1}{2a}\chi_{[-a,a]}\Big)^{\widehat{}}\;(\xi) \;=\; \frac{\sin(2\pi a)\xi}{(2\pi a)\xi}$$

By Fourier inversion, noting that $\frac{\sin x}{x}$ is not in L^1, only in L^2, so the inverse transform is not necessarily the literal integral,

$$\Big(\frac{\sin(2\pi a)\xi}{(2\pi a)\xi}\Big)^{\widehat{}}\;(x) \;=\; \frac{1}{2a}\chi_{[-a,a]}(x)$$

Replacing a by $a/2\pi$ gives

$$\Big(\frac{\sin a\xi}{a\xi}\Big)^{\widehat{}}\;(x) \;=\; \frac{\pi}{a}\,\chi_{[-\frac{a}{2\pi},\,\frac{a}{2\pi}]}(x)$$

We will use $a = 1, \frac{1}{3}$, and $\frac{1}{5}$. The relevant convolution was also computed above, but all we need is the fact that the support of

$$3\pi\,\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}$$

is inside the interval $[-\frac{1}{6\pi} - \frac{1}{10\pi},\ \frac{1}{6\pi} + \frac{1}{10\pi}]$. Thus, the integral of three *sinc* functions is equal to

$$\int_{\mathbb{R}} \pi\chi_{[\frac{-1}{2\pi},\frac{1}{2\pi}]}(x)\cdot\Big(3\pi\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * 5\pi\,\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\;dx \;=\; \pi\cdot 3\pi\cdot 5\pi\int_{-1/\pi}^{1/\pi}\Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\;dx$$

$$=\; \pi\cdot 3\pi\cdot 5\pi\int_{\mathbb{R}}\Big(\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]} * \chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]}\Big)(x)\;dx$$

since $[-1/2\pi,\ 1/2\pi]$ contains the support of the convolution. Observing that (invoking Fubini-Tonelli as necessary),

$$\int_{\mathbb{R}}(f*g)(x)\;dx \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x-y)g(y)\;dx\,dy \;=\; \int_{\mathbb{R}}\int_{\mathbb{R}} f(x)g(y)\;dx\,dy \;=\; \int_{\mathbb{R}} f(x)\;dx\cdot\int_{\mathbb{R}} g(x)\;dy$$

the integral of the convolution is

$$\int_{\mathbb{R}}\chi_{[-\frac{1}{6\pi},\,\frac{1}{6\pi}]}\cdot\int_{\mathbb{R}}\chi_{[-\frac{1}{10\pi},\,\frac{1}{10\pi}]} \;=\; \frac{1}{3\pi}\cdot\frac{1}{5\pi}$$

Thus, the whole is

$$\pi\cdot 3\pi\cdot 5\pi\cdot\frac{1}{3\pi}\cdot\frac{1}{5\pi} \;=\; \pi$$

Similarly, the integral of $f_1 * \ldots f_n$ is the product of the integrals $\int f_i$. With the support of f_i inside $[-a_i, a_i]$, the support of the convolution is inside $[-a_1 - \ldots - a_n,\ a_1 + \ldots + a_n]$. Thus, since $\frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{13} < 1$, the same argument shows that

$$\int_{\mathbb{R}}\frac{\sin x}{x}\cdot\frac{\sin x/3}{x/3}\cdot\frac{\sin x/5}{x/2n+1}\;dx \;=\; \pi \qquad\qquad (\text{for } 2n+1 = 3,5,7,9,11,13)$$

but for $2n+1 = 15$, the support of the Fourier transform of $\frac{\sin x}{x}$ no longer contains the support of the convolution. ///

[04.6] Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you assert: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$.)

Discussion: The idea is to invoke $f * g = (\widehat{f}\cdot\widehat{g})^{\widehat{}}$ for *even* functions $f, g \in L^1$, since for even functions the inverse Fourier transform is the same as the forward Fourier transform. Conveniently, Gaussians are in $L^1 \cap L^2$, and, from above, have Fourier transforms which are again Gaussians:

$$\widehat{e^{-\pi a x^2}}(\xi) \;=\; \frac{1}{\sqrt{a}}e^{-\pi\xi^2/a} \qquad\qquad (\text{for } a > 0)$$

so

$$e^{-\pi x^2} * e^{-\pi x^2}(\xi) \;=\; \widehat{e^{-\pi x^2} \cdot e^{-\pi x^2}}(\xi) \;=\; \widehat{e^{-2\pi x^2}}(\xi) \;=\; \frac{1}{\sqrt{2}} e^{-\pi \xi^2/2}$$

For the other example, the bound $|f * g|_{L^1} \le |f|_{L^p} \cdot |g|_{L^q}$ for conjugate exponents p, q shows that $f * g \in L^1$ for $f, g \in L^2$. Thus, the same identity holds for $f, g \in L^2$, with the Plancherel extension of Fourier transform. That is, $\widehat{f}$ and $\widehat{g}$ need not be the literal integrals for the Fourier transform, but its extension by continuity to L^2. Above, we computed the Fourier transform of characteristic functions of intervals:

$$\chi_{\widehat{[-a,a]}} a(\xi) \;=\; \frac{\sin 2\pi a \xi}{\pi \xi}$$

Thus,

$$\left(\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]}\right)\widehat{}\,(\xi) \;=\; \frac{\sin \xi}{\xi}$$

Then

$$\begin{aligned}\Big(\frac{\sin x}{x} * \frac{\sin x}{x}\Big)(\xi) \;&=\; \Big(\big(\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]}\big) \cdot \big(\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]}\big)\Big)\widehat{}\,(\xi) \\ &=\; \pi \cdot \big(\pi \cdot \chi_{[-1/2\pi,\, 1/2\pi]}\big)\widehat{}\,(\xi) \;=\; \pi \cdot \frac{\sin \xi}{\xi}\end{aligned}$$

///

[04.7] Prove that every $f \in C^o_c(\mathbb{R})$ can be uniformly approximated (in sup norm) arbitrarily well as superpositions of Gaussians: given $\varepsilon > 0$, there is $g \in C^o_c(\mathbb{R})$ and sufficiently large n such that

$$\sup_{x \in \mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} g(\xi) \cdot n e^{-\pi n^2 (\xi - x)^2} \, d\xi \Big| \;<\; \varepsilon$$

Discussion: This is an instance of an *approximate identity* and the basic property of such. Namely, for an approximate identity $\{\varphi_n\}$ on $\mathbb{R}$ and $f \in C^o_c(\mathbb{R})$, we have

$$\sup_{x \in \mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi_n(\xi) \cdot f(x + \xi) \, d\xi \Big| \;\longrightarrow\; 0 \qquad \text{(as } n \to +\infty)$$

By replacing ξ by $\xi - x$ in the integral, we have

$$\sup_{x \in \mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} f(\xi) \cdot \varphi_n(\xi - x) \, d\xi \Big| \;\longrightarrow\; 0 \qquad \text{(as } n \to +\infty)$$

Rather than reproving this general assertion in the example at hand, we simply clarify the interpretation in terms of approximate identities. That is, with $\varphi_1(x) = e^{-\pi x^2}$, we that the sequence $\varphi_n(x) = n \cdot \varphi_1(nx)$ is an approximate identity. More generally, we prove

[0.1] Claim: Let $\varphi \in C^o(\mathbb{R})$ be a non-negative $\mathbb{R}$-valued function, with $\int_{\mathbb{R}} \varphi = 1$. Then $\varphi_n(x) = n \cdot \varphi(n \cdot x)$ is an approximate identity.

Proof: The non-negative real-valued-ness is of course immediate. The integral of φ_n is

$$\int_{\mathbb{R}} \varphi_n(x) \, dx \;=\; \int_{\mathbb{R}} n \cdot \varphi(n \cdot x) \, dx \;=\; \int_{\mathbb{R}} n \cdot \varphi(x) \, \frac{dx}{n} \;=\; \int_{\mathbb{R}} \varphi(x) \, dx \;=\; 1$$

by replacing x by x/n in the integral. Finally, to see that the masses of the φ_n bunch up near 0: Since $\varphi \ge 0$ and

$$\lim_n \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x) \, dx \;=\; \int_{\mathbb{R}} \varphi(x) \, dx \;=\; 1$$

given $\varepsilon > 0$ there is sufficiently large n_o such that for all $n \ge n_o$

$$1 \;\le\; \lim_n \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x)\, dx \;>\; 1-\varepsilon$$

Then, by replacing x by x/n in the integral,

$$\int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} \varphi_n(x)\, dx \;=\; \int_{-\frac{1}{\sqrt{n}}}^{\frac{1}{\sqrt{n}}} n\cdot\varphi(n\cdot x)\, dx \;=\; \int_{-\sqrt{n}}^{\sqrt{n}} \varphi(x)\, dx \;>\; 1-\varepsilon$$

The verifies the bunching-up property. ///

[04.8] Without worrying too much about identifying the finite, positive constant $\int_{\mathbb{R}} \frac{(\sin x)^2}{x^2}\, dx$, prove that, for given $f \in C_c^o(\mathbb{R})$, given $\varepsilon > 0$, there is sufficiently large n and a function $g \in C_c^o(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} g(\xi)\cdot \frac{(\sin n(x-\xi))^2}{(x-\xi)^2}\, d\xi \Big| \;<\; \varepsilon$$

Discussion: After the more general discussion of the previous example, this is just another such. ///

[04.9] Show that the *principal value* functional

$$f \;\longrightarrow\; PV\int_{\mathbb{R}} \frac{f(x)}{x}\, dx \;=\; \lim_{\varepsilon\to 0}\Big(\int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x}\, dx\Big)$$

is equal to

$$-\int_{\mathbb{R}} f'(x)\cdot \log|x|\, dx$$

for f *continuously* differentiable on $\mathbb{R}$, with hypotheses on the decay of f and f' at infinity.

Discussion: This is an exercise in careful integration by parts, in the course of which we discover reasonable hypotheses on f and f' so that the natural heuristic is a proof.

For fixed small $\varepsilon > 0$ and large $M > 0$, integration by parts gives

$$\int_{-M}^{-\varepsilon} \frac{f(x)}{x}\, dx + \int_{\varepsilon}^{M} \frac{f(x)}{x}\, dx$$

$$= \Big[f(x)\cdot\log|x|\Big]_{-M}^{-\varepsilon} + \Big[f(x)\cdot\log x\Big]_{\varepsilon}^{M} - \int_{-M}^{-\varepsilon} f'(x)\cdot\log|x|\, dx - \int_{\varepsilon}^{M} f'(x)\cdot \log x\, dx$$

The simplest way to make the boundary terms near $\pm\infty$ vanish is that they vanish *individually.* For example, it does not suffice that $f \in L^1(\mathbb{R})$ or $L^2(\mathbb{R})$, because such a hypothesis by itself does not assure that $f(x)\cdot\log|x|$ goes to 0 at $\pm\infty$, since f could have narrower-and-narrower spikes parading out to infinity. It is true that an additional condition on the derivative might promise this asymptotic behavior of f, but let's not be toooo clever. So, exactly require that $f(x)\cdot\log|x|$ goes to 0 at $\pm\infty$. In contrast, making

$$\lim_{\varepsilon\to 0^+}\Big(f(\varepsilon)\cdot\log|\varepsilon| - f(-\varepsilon)\cdot\log|\varepsilon|\Big) \;=\; 0$$

is the most subtle issue here. It cannot reasonably accomplished by having the individuals go to 0, unless we require some sort of vanishing of f at 0, which would be undesirable here. Here is where *differentiability* of

f at 0 can be used: a Taylor-Maclaurin expansion with remainder ensures that $f(\varepsilon) - f(-\varepsilon) = O(\varepsilon)$. Since $\lim_{\varepsilon\to 0^+} \varepsilon \log \varepsilon = 0$, the combination of these boundary terms does go to zero.

For the limit as $M \to +\infty$ of the individual integrals of $f'(x) \cdot \log|x|$ to exist in a simple fashion, it suffices that $\lim_{M\to\infty} \int_M^\infty f'(x) \cdot \log|x|\, dx = 0$ and $\lim_{M\to\infty} \int_{-\infty}^{-M} f'(x) \cdot \log|x|\, dx = 0$. The last question is about what it takes to make

$$\lim_{\varepsilon\to 0^+} \Big(\int_\varepsilon^\infty f'(x) \cdot \log|x|\, dx + \int_{-\infty}^{-\varepsilon} f'(x) \cdot \log|x|\, dx \Big) \;=\; \int_{\mathbb{R}} f'(x) \cdot \log|x|\, dx$$

Since $\log|x|$ is locally integrable, for f' being merely *essentially bounded* on some interval $[-\varepsilon_o, \varepsilon_o]$, e.g., *continuous*, the two individual integrals $\int_{\varepsilon'}^{\varepsilon} f'(x) \cdot \log|x|\, dx$ and $\int_{-\varepsilon}^{-\varepsilon'} f'(x) \cdot \log|x|\, dx$ for $0 < \varepsilon' < \varepsilon$ go to zero as $\varepsilon \to 0^+$. Thus, with these natural sufficient constraints, we have the indicated identity. ///

[04.10] Let $\psi_n(x) = e^{2\pi i n x}$. Let $\delta_{\mathbb{Z}}$ be the *Dirac comb*, that is, a periodic version of Dirac's δ, describable as having Fourier series

$$\delta_{\mathbb{Z}} \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \qquad\qquad \text{(converging in } H^{-1}(\mathbb{T}) \text{ or even } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon > 0)$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda \cdot u \;=\; \delta_{\mathbb{Z}}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division.* Show that the equation $v'' - \lambda v = f$ is solved by

$$v(x) \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_0^1 u(x-t)\, f(t)\, dt$$

Discussion: Using the *spectral* characterization of the $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ norm,

$$\Big| \sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \Big|^2_{H^{-\frac{1}{2}-\varepsilon}} \;=\; \sum_{n\in\mathbb{Z}} |1|^2 \cdot (1+n^2)^{-\frac{1}{2}-\varepsilon}$$

which is convergent for all $\varepsilon > 0$, by comparison to $\sum_{n\neq 0} 1/n^2$. So that Fourier series converges in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ and produces a *generalized function* there.

The extension by continuity of d/dx from $C^\infty(\mathbb{T}) \to C^\infty(\mathbb{T})$ to $\widetilde{\frac{d}{dx}} : H^s(\mathbb{T}) \to H^{s-1}(\mathbb{T})$ is continuous, by design. Similarly, $\widetilde{\frac{d^2}{dx^2}} : H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ is continuous. That is, since infinite sums are the corresponding limits of finite partial sums, this continuity means that termwise differentiation is correct. Let $u = \sum_n c_n \psi_n$, and solve, dropping the tilde from the notation,

$$\sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \;=\; \delta_{\mathbb{Z}} \;=\; u'' - \lambda u \;=\; \sum_n c_n \Big(\frac{d^2}{dx^2} - \lambda \Big) \psi_n \;=\; \sum_n c_n (-4\pi^2 n^2 - \lambda) \cdot \psi_n$$

Equating coefficients, $c_n = 1/(-4\pi^2 n^2 - \lambda)$, for λ not equal to $-4\pi^2 n^2$ for integer n. Another easy estimate shows that this u has gained 2 Sobolev indices, so is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{T})$.

By Sobolev imbedding/inequality, $H^s(\mathbb{T}) \subset C^o(\mathbb{T})$ for all $s > \frac{1}{2}$, so the solution is continuous (and, in fact, satisfies a further Lipschitz condition).

To see that $v'' - \lambda v = f$ is solved by

$$v(x) \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\; dt \;=\; \int_0^1 u(x-t)\, f(t)\; dt$$

take f such that $\widehat{f} \in \ell^1_{\mathbb{Z}}(\mathbb{T})$, meaning that $\sum_n |\widehat{f}(n)| < \infty$. A somewhat stronger, more intuitive assumption is that $f \in C^2(\mathbb{T})$, and then by integration by parts

$$\widehat{f''}(n) \;=\; \int_{\mathbb{T}} e^{-2\pi i n x}\, f''(x)\; dx \;=\; \int_{\mathbb{T}} (-2\pi i n)^2\, e^{-2\pi i n x}\, f(x)\; dx \;=\; (2\pi i n)^2 \cdot \widehat{f}(n)$$

(On the circle $\mathbb{T}$, and/or for $\mathbb{Z}$-periodic functions, there are no boundary terms in integration by parts.) We do not even to invoke Riemann-Lebesgue, since $|\widehat{f''}(n)|$ is *bounded*, so there is a constant C such that $|\widehat{f}(n)| \le C/n^2$, so $\widehat{f} \in \ell^1_{\mathbb{Z}}$.

Then Fubini-Tonelli assures the legitimacy of interchanging sum and limit: [1]

$$\int_{\mathbb{T}} u(x-t)\, f(t)\; dt \;=\; \int_{\mathbb{T}} \sum_m \frac{1}{-4\pi^2 m^2 - \lambda} e^{2\pi i m(x-t)} \cdot \sum_n \widehat{f}(n) e^{2\pi i n t}\; dt$$

$$= \sum_{m,n} \frac{\widehat{f}(n)}{-4\pi^2 m^2 - \lambda} \int_{\mathbb{T}} e^{2\pi i m(x-t)}\, e^{2\pi i n t}\; dt \;=\; \sum_n \frac{\widehat{f}(n)}{-4\pi^2 n^2 - \lambda} e^{2\pi i n x}$$

by mutual orthogonality of distinct exponentials (in every Sobolev space). By Riemann-Lebesgue, $\widehat{f}(n) \to 0$, so

$$\sum_n |\widehat{f}(n)|^2 \cdot (1+n^2)^s \;<\; \infty \qquad\qquad \text{(for any } s < -\tfrac{1}{2}\text{)}$$

so $f \in H^{-1}(\mathbb{T})$, for example. Application of the (extension of) $\frac{d^2}{dx^2} - \lambda$ *termwise* (again, justified by continuity of the extension) produces the Fourier expansion of f. ///

[0.2] Remark: The latter example illustrates the utility of using *generalized functions* even in a discussion that seems not to refer to them: there was no need to *guess* the function $u(x-t)$ (sometimes called a *Green's function*) solving the differential equation, since we *solved* for it using the Fourier expansion of $\delta_{\mathbb{Z}}$ that only converges in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$.

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[1] In fact, we will see later that for u continuous and f in any Sobolev space, the interchange is justified.