(*December 18, 2017*)

Examples discussion 05

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

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[05.1] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this kind of construction to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0,1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1,2,3,\ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n,n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1),-n]$ and monotonically from 1 to 0 in $[n, n+1]$.

Discussion: In $x > 0$, by induction, the derivatives are finite linear linear combinations of functions of the form $x^{-n}e^{-1/x}$. It suffices to show that $\lim_{x\to 0^+} x^{-n}e^{-1/x} = 0$. Equivalently, that $\lim_{x\to+\infty} x^n e^{-x} = 0$, which follows from $e^{-x} = 1/e^x$, and

$$x^{-n}e^{-1/x} \;=\; \frac{x^n}{e^x} \;=\; \frac{x^n}{\sum_{m\ge 0}\frac{x^m}{m!}} \;\le\; \frac{x^n}{\frac{x^{n+1}}{(n+1)!}} \;\longrightarrow\; 0 \qquad\qquad (\text{as } x \to +\infty)$$

(This is perhaps a little better than appeals to L'Hospital's Rule.) Thus, f is smooth at 0, with all derivatives 0 there. ///

Next, we make a *smooth bump function* by

$$b(x) \;=\; \begin{cases} 0 & (\text{for } x \le -1) \\ \\ e^{\frac{1}{1-x^2}} & (\text{for } -1 < x < 1) \\ \\ 0 & (\text{for } x \ge 1) \end{cases}$$

A similar argument to the previous shows that this is smooth. Renormalize it to have integral 1 by

$$\beta(x) \;=\; \frac{b(x)}{\int_{-1}^{1} b(t)\; dt}$$

Then $\int_{-1}^{x}\beta(t)\;dt$ is a smooth (monotone) step function that goes from 0 at -1 to 1 at 1. The minor modification $s(x) = 2\int_{-1}^{x}\beta(2t-1)\;dt$ gives a smooth (monotone) step function going from 0 at 0 to 1 at 1. ///

Then $s(x+n+1)$ is a smooth, monotone step function going up from 0 to 1 in $[-n-1,-n]$, and $s(n+1-x)$ for $n \in \mathbb{Z}$ is a smooth, monotone step function going *down* from 1 to 0 in $[n, n+1]$. Thus, the product $f_n(x) = s(x+n+1)\cdot s(n+1-x)$ is the desired smooth cut-off function. ///

[05.2] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, first for $f \in \mathscr{S}(\mathbb{R}^n)$, then for $f \in \mathscr{S}(\mathbb{R}^n)^*$.

Discussion: For $f \in \mathscr{S}(\mathbb{R}^n)$, the literal integral computes the Fourier transform:

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\; g(x)\; dxn \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\; f(x+x_o)\; dx$$

Replacing x by $x - x_o$ in the integral gives

$$\widehat{g}(\xi) \;=\; \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot(x-x_o)}\, f(x)\,dx \;=\; e^{2\pi i\xi\cdot x_o}\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}\, f(x)\,dx \;=\; e^{2\pi i\xi\cdot x_o}\cdot \widehat{f}(\xi)$$

The precise corresponding statement for tempered distributions cannot refer to pointwise values. Write ψ_{x_o} for the function $\xi \to e^{2\pi i\xi\cdot x_o}$. Since ψ_{x_o} is bounded, for a tempered distribution u, $\psi_{x_o}\cdot u$ is the tempered distribution described by

$$(\psi_{x_o}\cdot u)(\varphi) \;=\; u(\psi_{x_o}\,\varphi) \qquad\qquad (\text{for } \varphi\in\mathscr{S})$$

This is compatible with multiplication of (integrate-against-) functions $\mathscr{S}\subset\mathscr{S}^*$. Also, let translation $u\to T_{x_o}u$ be defined by $(T_{x_o}u)(\varphi) = u(T_{-x_o}\varphi)$, again compatibly with integration against Schwartz functions. In these terms, the above argument shows that

$$(T_{x_o}f)\widehat{\ } = \;\psi_{x_o}\cdot\widehat{f} \qquad\qquad (\text{for } f\in\mathscr{S})$$

This formulation avoids reference to pointwise values, and thus could make sense for tempered distributions.

One argument is *extension by continuity*: Fourier transform is a continuous map $\mathscr{S}^*\to\mathscr{S}^*$, as is translation $u\to T_{x_o}u$, so the identity extends by continuity to all tempered distributions. ///

Another argument is by *duality*: first,

$$(T_{x_o}u)\widehat{\ }(\varphi) \;=\; (T_{x_o}u)(\widehat{\varphi}) \;=\; u(T_{-x_o}\,\widehat{\varphi}) \;=\; u\Big((\psi_{x_o}\cdot\varphi)\widehat{\ }\Big)$$

by applying the identity to $\varphi,\widehat{\varphi}\in\mathscr{S}$. Going back, this is

$$\widehat{u}(\psi_{x_o}\cdot\varphi) \;=\; (\psi_{x_o}\cdot\widehat{u})(\varphi) \qquad\qquad (\text{for all } \varphi\in\mathscr{S})$$

Altogether, $(T_{x_o}u)\widehat{\ } = \;\psi_{x_o}\cdot\widehat{u}$.

[05.3] Let V be a vector space, with norms $|\cdot|_1$ and $|\cdot|_2$. Suppose that $|v|_2\ge|v|_1$ for all $v\in V$. Show that the identity map $i:V\to V$ is continuous, where the source is given the $|\cdot|_2$ topology and the target is given the $|\cdot|_1$ topology. Show that if a sequence $\{v_n\}$ in V is $|\cdot|_2$ Cauchy, then it is $|\cdot|_1$-Cauchy. Let V_j be the completion of V with respect to the metric $|v-v'|_j$. Show that we can *extend* i *by continuity* to a continuous linear map $I:V_2\to V_1$, that is, by

$$I(V_2\text{-limit of } V_2\text{-Cauchy sequence } \{v_n\}) \;=\; V_1\text{-limit of } \{v_n\}$$

Discussion: First, it suffices to show that the identity map $i:V\to V$ with indicated topologies is *bounded*, and, indeed,

$$|j(v)|_1 \;=\; |v|_1 \;\le\; |v|_2 \qquad\qquad (\text{for all } v\in V, \text{ by hypothesis})$$

For $\{v_n\}$ Cauchy in the $|\cdot|_2$ topology, given $\varepsilon>0$, take n_o such that $|v_m-v_n|_2<\varepsilon$ for all $m,n\ge n_o$. Then the same inequality holds (with the same n_o and ε) for $|\cdot|_2$, so $\{v_n\}$ is Cauchy in the $|\cdot|_1$ topology.

A useful characterization of the completion $\widetilde{X}$ of a metric space X is that there is an isometry $j:X\to\widetilde{X}$, and any non-expanding[1] map $f:X\to Y$ to a complete metric space Y extends uniquely to continuous map $\widetilde{f}:\widetilde{X}\to Y$, with $\widetilde{f}\circ j = f$. In particular,

$$\widetilde{f}(X-\lim_n x_n) \;=\; Y-\lim_n f(x_n)$$

[1] This sense of *non-expanding* is the reasonable one: $d_Y(f(x),f(x'))\le d_X(x,x')$ for all $x,x'\in X$.

This is well-defined because f is continuous on X. Thus, with $X = V$, $\widetilde{X} = V_2$, $Y = V_1$, and $f : V \to V_1$ given by inclusion, we have the assertion. ///

[05.4] Solve $-u'' + u = \delta$ on $\mathbb{R}$. (*Hint:* use Fourier transform, and grant that $\widehat{\delta} = 1$.)

Discussion: Let's assume that we are asking for a solution u that is at worst a tempered distribution. Thus, we can take Fourier transform, obtaining

$$(4\pi^2\xi^2 + 1)\widehat{u} \;=\; \widehat{\delta} \;=\; 1$$

Obviously we want to *divide* by $4\pi^2\xi^2 + 1$. Unlike some other examples, where division was not quite legitimate, here, we can achieve the effect by *multiplication* by the smooth, bounded function $1/(4\pi^2\xi^2 + 1)$, since $4\pi^2\xi^2 + 1$ does not vanish on $\mathbb{R}$. Thus,

$$\widehat{u} \;=\; \frac{1}{4\pi^2\xi^2 + 1}$$

Since the right-hand side is luckily in $L^1(\mathbb{R})$, we can compute its image under Fourier inversion by the literal integral, its inverse Fourier transform will be a continuous function (by Riemann-Lebesgue), so has meaningful pointwise values:

$$u(x) \;=\; \int_{\mathbb{R}} \frac{e^{2\pi i\xi x}}{4\pi^2\xi^2 + 1}\, d\xi$$

The integral can be evaluated by *residues*: depending on the sign of x, we use an auxiliary arc in the upper (for $x > 0$) or lower (for $x < 0$) half-plane, so that $\xi \to e^{2\pi i\xi x}$ is *bounded* in the corresponding half-plane. Thus, we pick up either $2\pi i$ times the residue at $\xi = 1/2\pi i$, or the negative (because the orientation is negative) of the residue at $\xi = -1/2\pi i$. That is, respectively,

$$2\pi i \cdot \frac{e^{2\pi i\cdot(1/2\pi i)\cdot x}}{4\pi^2\cdot(\frac{1}{2\pi i} - \frac{-1}{2\pi i})} \;=\; \frac{-e^{-x}}{2} \;=\; \frac{-e^{-|x|}}{2} \qquad\qquad (\text{for } x \ge 0)$$

and

$$-2\pi i \cdot \frac{e^{2\pi i\cdot(-1/2\pi i)\cdot x}}{4\pi^2\cdot(\frac{-1}{2\pi i} - \frac{1}{2\pi i})} \;=\; \frac{-e^{x}}{2} \;=\; \frac{-e^{-|x|}}{2} \qquad\qquad (\text{for } x \le 0)$$

[05.5] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series, granting the Fourier expansion of $\delta_{\mathbb{Z}}$.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution.

Discussion: In Fourier series converging in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{T})$ for all $\varepsilon > 0$, $\delta_{\mathbb{Z}} = \sum_{n\in\mathbb{Z}} 1\cdot\psi_n$, where $\psi_n(x) = e^{2\pi inx}$. A function u in the relatively large-yet-tractable space $H^{-\infty}(\mathbb{T})$ has a Fourier expansion $u = \sum_n \widehat{u}(n)\cdot\psi_n$. Application of the (extended-sense) second derivative operator can be done termwise (by design), and annihilates the $n = 0$ term. That is, no u'' can have 0^{th} Fourier coefficient 1, as does $\delta_{\mathbb{Z}}$, so that equation is not solvable. ///

In contrast, $\delta_{\mathbb{Z}} - 1$ has exactly lost that difficult Fourier component, and, in terms of Fourier series, $u'' = \delta_{\mathbb{Z}} - 1$ is

$$\sum_{n\in\mathbb{Z}} (2\pi in)^2)\cdot\widehat{u}(n)\cdot\psi_n \;=\; \sum_{n\ne 0} 1\cdot\psi_n$$

has the solution *by division*

$$u \;=\; \sum_{n\ne 0} \frac{1}{(2\pi in)^2}\psi_n$$

[05.6] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1. (And reflect on the Fredholm alternative?)

Discussion: The orthogonality to 1 means that the 0^{th} Fourier coefficient of f is 0. Thus, on the Fourier series side, for any $u \in H^{-\infty}(\mathbb{T})$, $u'' = f$ is

$$\sum_{n\in\mathbb{Z}} (2\pi i n)^2 \cdot \widehat{u}(n) \cdot \psi_n \;=\; \sum_{n\neq 0} \widehat{f}(n) \cdot \psi_n$$

gives

$$u \;=\; \sum_{n\neq 0} \frac{\widehat{f}(n)}{(2\pi i n)^2} \cdot \psi_n$$

and there is no other solution in $H^{-\infty}(\mathbb{T})$. ///

[05.7] The sawtooth function is first defined on $[0,1)$ by $\sigma(x) = x - \frac{1}{2}$, and then extended to $\mathbb{R}$ by periodicity so that $\sigma(x+n) = \sigma(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. After recalling its Fourier expansion, describe the derivatives σ' and σ'' of σ.

Discussion: The 0^{th} Fourier coefficient is 0. For $n \neq 0$, integrating by parts once, the n^{th} Fourier is $-1/2\pi i n$. That is, at least converging in L^2,

$$\sigma(x) \;=\; \sum_{n\neq 0} \frac{1}{-2\pi i n}\, e^{2\pi i n x}$$

In fact, from the Fourier-coefficient criterion for Sobolev spaces, $\sigma \in H^{\frac{1}{2}-\varepsilon}$ for all $\varepsilon > 0$. Differentiating termwise (in an extended sense),

$$\sigma' \;=\; -\sum_{n\neq 0} e^{2\pi i n x} \qquad\qquad (\text{convergent in } H^{-\frac{1}{2}-\varepsilon} \text{ for all } \varepsilon > 0)$$

We might recognize this as being closely related to the Dirac comb

$$\delta_{\mathbb{Z}} \;=\; \sum_{n\in\mathbb{Z}} e^{2\pi i n x} \qquad\qquad (\text{convergent in } H^{-\frac{1}{2}-\varepsilon})$$

Specifically, $\sigma' = 1 - \delta_{\mathbb{Z}}$. Also, looking at the description of σ directly, its derivative is (locally) 1 away from $\mathbb{Z}$, and has a $-\delta_n$ for all $n \in \mathbb{Z}$. That is, yet again,

$$\sigma' \;=\; 1 - \sum_{n\in\mathbb{Z}} \delta_n \;=\; 1 - \delta_{\mathbb{Z}}$$

Similarly, differentiating term-wise once more,

$$\sigma'' \;=\; -\sum_{n\neq 0} 2\pi i n \cdot e^{2\pi i n x} \qquad\qquad (\text{convergent in } H^{-\frac{3}{2}-\varepsilon} \text{ for all } \varepsilon > 0)$$

$$=\; -\sum_{n\in\mathbb{Z}} \delta'_n \;=\; -\delta'_{\mathbb{Z}}$$

///

[05.8] Show that $e^{-\varepsilon\pi x^2} \to 1$ as $\varepsilon \to 0^+$ in the $\mathscr{S}^*$ topology. Compute the Fourier transforms of the functions $e^{-\varepsilon\pi x^2}$, and show that they go to δ in the $\mathscr{S}^*$ topology. Obtain, again, as a corollary, the fact that $\widehat{1} = \delta$ (extended Fourier transform).

Discussion: We must show that, for each $\varphi \in \mathscr{S}$,

$$\lim_{\varepsilon\to 0^+} \int_{\mathbb{R}} (e^{-\varepsilon\pi x^2} - 1)\,\varphi(x)\,dx \;=\; 0$$

Since we are accustomed to other uses of ε, let's rewrite this as

$$\lim_{\eta\to 0^+} \int_{\mathbb{R}} (e^{-\eta\pi x^2} - 1)\,\varphi(x)\,dx \;=\; 0$$

Given $\varepsilon > 0$, for given φ, let N be sufficiently large so that $|\varphi(x)| < \varepsilon/|x|^2$ for $|x| \geq N$. Then

$$\Big|\int_{\mathbb{R}} (e^{-\eta\pi x^2} - 1)\,\varphi(x)\,dx\Big| \;\leq\; \int_{|x|\leq N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\,dx + \int_{|x|\geq N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\,dx$$

Estimate the second integral:

$$\int_{|x|\geq N} |e^{-\eta\pi x^2} - 1|\cdot|\varphi(x)|\,dx \;\leq\; \int_{|x|\geq N} 2\cdot\frac{\varepsilon}{|x|^2}\,dx \;\leq\; \frac{4}{N}\cdot\varepsilon$$

For the first integral, given N and ε, for sufficiently small $\eta > 0$, we have $|e^{-\eta\pi x^2} - 1| < \varepsilon$ for all $|x| \leq N$. Thus, $e^{-\eta\pi x^2} \to 1$. ///

Next, the usual trick computes the Fourier transform

$$\widehat{e^{-\eta\pi x^2}}(\xi) \;=\; \frac{1}{\sqrt{\eta}}\cdot e^{-\frac{1}{\eta}\pi\xi^2}$$

We want to show that these go to δ. The continuity of (extended) Fourier transform $\mathscr{S}^* \longrightarrow \mathscr{S}^*$ assures that

$$(\mathscr{S}^*\text{–}\lim_{\eta} e^{-\eta\pi x^2})\widehat{\ } \;=\; \mathscr{S}^*\text{–}\lim_{\eta} \widehat{e^{-\eta\pi x^2}} \;=\; \mathscr{S}^*\text{–}\lim_{\eta} \frac{1}{\sqrt{\eta}} e^{\widehat{-\frac{1}{\eta}\pi x^2}}$$

For each $\varphi \in \mathscr{S}$,

$$\int_{\mathbb{R}} \varphi(x)\cdot\frac{1}{\sqrt{\eta}}\cdot e^{-\frac{1}{\eta}\pi x^2}\,dx \;=\; \int_{\mathbb{R}} \varphi(\sqrt{\eta}\cdot x)\cdot e^{-\pi x^2}\,dx$$

by replacing x by $\sqrt{\eta}\cdot x$. Given $\varepsilon > 0$, let N be large enough so that $|e^{-\pi x^2}| < \varepsilon$ for $|x| \geq N$. Let $\delta > 0$ be small enough so that $|\varphi(x) - \varphi(0)| < \varepsilon$ for $|x| < \delta$. Take $\eta > 0$ sufficiently small so that $N\cdot\sqrt{\eta} < \delta$. Using $\int_{\mathbb{R}} e^{-\pi x^2}\,dx = 1$,

$$\Big|\varphi(0) - \int_{\mathbb{R}} \varphi(\sqrt{\eta}\cdot x)\cdot e^{-\pi x^2}\,dx\Big| \;=\; \Big|\int_{\mathbb{R}} (\varphi(0) - \varphi(\sqrt{\eta}\cdot x))\cdot e^{-\pi x^2}\,dx\Big|$$

$$\leq\; \int_{|x|\leq N} |\varphi(0) - \varphi(\sqrt{\eta}\cdot x)|\cdot e^{-\pi x^2}\,dx + \int_{|x|\geq N} |\varphi(0) - \varphi(\sqrt{\eta}\cdot x)|\cdot e^{-\pi x^2}\,dx$$

$$<\; \int_{|x|\leq N} \varepsilon\cdot e^{-\pi x^2}\,dx + \int_{|x|\geq N} 2\sup|\varphi|\cdot e^{-\pi x^2}\,dx \;<\; \varepsilon + 2\sup|\varphi|\cdot\varepsilon$$

That is, $\frac{1}{\sqrt{\eta}}\cdot e^{-\frac{1}{\eta}\pi x^2} \longrightarrow \delta$ in the $\mathscr{S}^*$ topology. ///

[05.9] Compute $\widehat{\cos x}$.

Discussion: Start from $\widehat{\delta} = 1$. Using the previous example's identity,

$$(T_{x_o}\delta)\widehat{\ } \;=\; \psi_{x_o}\cdot 1 \;=\; \psi_{x_o}$$

By Fourier inversion, $\widehat{\psi_{x_o}} = T_{-x_o}\delta$. Thus,

$$\widehat{\cos x} \;=\; \tfrac{1}{2}(\psi_{1/2\pi} + \psi_{-1/2\pi})\widehat{\ } \;=\; \tfrac{1}{2}(T_{-1/2\pi}\delta + T_{1/2\pi}\delta)$$

Written in terms of mock-pointwise-values, this is $\widehat{\cos}(\xi) = \dfrac{\delta(\xi - \frac{1}{2\pi}) + \delta(\xi + \frac{1}{2\pi})}{2}$. ///

[05.10] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\mathrm{spt}\,\varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x \cdot u = 0$ implies that u is a multiple of δ itself.

Discussion: For $\varphi \in \mathcal{D}$ whose support does *not* include 0, the function $1/x$ is defined and smooth on $\mathrm{spt}\,\varphi$. Thus, $x \to \varphi(x)/x$ is in $\mathcal{D}$. For such φ,

$$u(\varphi) \;=\; u(x \cdot \frac{\varphi}{x}) \;=\; 0$$

Thus, $\mathrm{spt}\,u = \{0\}$, so by the theorem is a finite linear combination $u = \sum_{i=0}^n c_i\,\delta^{(i)}$ with scalars c_i. To see that in fact only δ itself can appear, we use the idea that $1, x, \frac{x^2}{2!}, \frac{x^3}{3!}, \ldots, \frac{x^n}{n!}$ are essentially a *dual basis* to $\delta, \delta', \delta'', \ldots, \delta^{(n)}$. One way to make this completely precise is to use a smooth cut-off function $\eta \in \mathcal{D}$ around 0, namely, identically 1 on a neighborhood of 0. Then $\eta \cdot x^i \in \mathcal{D}$, and

$$\delta^{(i)}(\eta \cdot \frac{x^j}{j!}) \;=\; \begin{cases} 1 & (\text{for } i = j) \\ \\ 0 & (\text{for } i \neq j) \end{cases}$$

In particular, this shows that the derivatives of δ are *linearly independent*. For $0 \le j \in \mathbb{Z}$,

$$0 \;=\; (x \cdot u)(x^j) \;=\; (x \cdot \sum_i c_i\,\delta^{(i)})(x^j) \;=\; \sum_i c_i\,\delta^{(i)}(x \cdot x^j) \;=\; \sum_i c_i\,\delta^{(i)}(x^{j+1}) \;=\; (j+1)! \cdot c_{j+1}$$

Thus, $c_j = 0$ for $j \ge 1$, and u is a multiple of δ itself. ///