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Examples discussion 06

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/˜garrett/

[This document is http://www.math.umn.edu/˜garrett/m/real/examples_2017-18/real-disc-06.pdf]

[06.1] Given f in the Schwartz space $\mathscr{S}$, show that there is $F \in \mathscr{S}$ with $F' = f$ if and only if $\int_{\mathbb{R}} f = 0$.

Discussion: On one hand, if $f = F'$ for $F \in \mathscr{S}$, then $\int_{-\infty}^x f(y)\,dy = F(x)$. Since $\lim_{x\to+\infty} F(x) = 0$, $\int_{\mathbb{R}} f = 0$.

On the other hand, if $\int_{\mathbb{R}} f = 0$, let $F(x) = \int_{-\infty}^x f$, and show that $F \in \mathscr{S}$. Since $F' = f$ by the fundamental theorem of calculus, the (higher) derivatives of F are those of f, so all that needs to be shown is that F itself is of rapid decay. For $x \to -\infty$,

$$|F(x)| \;\le\; \int_{-\infty}^x |f| \;\le\; \int_{-\infty}^x |1+y^2|^{-N}\cdot \sup_{t\in\mathbb{R}}\Big|(1+t^2)^N\cdot f(t)\Big|\,dy \;\le\; \sup_{t\in\mathbb{R}}\Big|(1+t^2)^N\cdot f(t)\Big|\cdot\int_{-\infty}^x |1+y^2|^{-N}\,dy$$

$$\le\; \sup_{t\in\mathbb{R}}\Big|(1+t^2)^N\cdot f(t)\Big|\cdot\int_{-\infty}^x \frac{dt}{t^N} \;\le\; \sup_{t\in\mathbb{R}}\Big|(1+t^2)^N\cdot f(t)\Big|\cdot\frac{1}{|x|^{N-1}} \qquad (\text{for } x\to-\infty)$$

giving the rapid decay. For $x \to +\infty$, using the condition $\int_{\mathbb{R}} f = 0$,

$$F(x) \;=\; \int_{-\infty}^x f \;=\; \int_{\mathbb{R}} f - \int_x^\infty f \;=\; 0 - \int_x^\infty f$$

so for $x \to +\infty$ it suffices to similarly estimate

$$\Big|\int_x^\infty f\Big| \;\le\; \int_x^\infty (1+y^2)^{-N}\cdot\sup_{t\in\mathbb{R}}|(1+t^2)^N\cdot f(t)|\,dy \;\le\; \sup_{t\in\mathbb{R}}|(1+t^2)^N\cdot f(t)|\cdot\int_x^\infty (1+y^2)^{-N}\,dy$$

which similarly gives the rapid decay as $x \to +\infty$. ///

[06.2] Let $u(x) = e^x\cdot\sin(e^x)$. Explain in what sense the integral $\displaystyle\int_{\mathbb{R}} f(x)\,u(x)\,dx$ converges for every $f \in \mathscr{S}$.

Discussion: The idea is to integrate by parts, noting that $u = v'$ with $v(x) = \cos(e^x)$. We must be careful with the boundary terms:

$$\int_{\mathbb{R}} f(x)\,u(x)\,dx \;=\; \int_{\mathbb{R}} f(x)\,v'(x)\,dx \;=\; \lim_{M,N\to+\infty}\int_{-M}^N f(x)\,v'(x)\,dx$$

$$=\; \lim_{M,N\to+\infty}\Big(\Big[f(x)\,v(x)\Big]_{-M}^N - \int_{-M}^N f'(x)\,v(x)\,dx\Big)$$

Since $v(x)$ is bounded and f' is of rapid decay, the limit *exists*, so the original integral is convergent. Further, the value is correctly determined by integration by parts, namely

$$-\int_{-\infty}^\infty f'(x)\,v(x)\,dx \;=\; -\int_{-\infty}^\infty f'(x)\,\cos(e^x)\,dx$$

That is, for $f \in \mathscr{S}$ and functions such as u obtained by differentiating bounded smooth functions, integration by parts is completely justifiable via the natural estimates. ///

[06.3] Show that $\sin(nx) \to 0$ in the $\mathscr{S}^*$-topology as $n \to +\infty$. (Since $\mathscr{S}$ is strictly larger than $\mathcal{D}$, this implies that $\sin(nx) \to 0$ in the $\mathcal{D}^*$-topology.)

Discussion: We must show that, for each $\varphi \in \mathscr{S}$,

$$\lim_n \int_{\mathbb{R}} \sin(nx)\, \varphi(x)\, dx \;=\; 0$$

On one hand, since Schwartz functions are L^1, we could invoke Riemann-Lebesgue, since (up to normalizations) the indicated integral is $(\widehat{\varphi}(n) - \widehat{\varphi}(-n))/2i$.

On another hand, we also know that $\widehat{\varphi}$ is again a Schwartz function, so $(\widehat{\varphi}(n) - \widehat{\varphi}(-n))/2i \to 0$. (Further, if we know that $\mathscr{S}$ is dense in L^1, then this gives a slightly different proof of Riemann-Lebesgue.) ///

[06.4] Let $-\infty < a < b < c < +\infty$, and

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x < a) \\ A & (\text{for } a < x < b) \\ B & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Show that (extended) $\frac{d}{dx} f = A\delta_a + (B - A)\delta_b - B\delta_c$.

Discussion: This example asks for *proof* of the plausible intuitive idea that a piecewise constant function has derivative 0 along the intervals where it is constant, and multiples of Dirac deltas where jumps occur. There are at least two approaches to the proof, depending whether one characterizes distributions as elements of a dual space, or as $\mathcal{D}^*$-limits of test functions. Granting the theorem that these two characterizations are equivalent, the operational question is which allows an easier approach to the present question.

Perhaps the characterization by duality is more convenient here. Thus, $f' \in \mathcal{D}^*$ is a linear functional on $\mathcal{D}$ characterized by the extension of integration by parts:

$$(\text{as functional})\ f'(\varphi) \;=\; -f(\varphi') \;=\; -\int_{\mathbb{R}} f(x)\, \varphi'(x)\, dx \qquad\qquad (\text{for all } \varphi \in \mathcal{D})$$

Yes, the notation is slightly inconsistent, since on the left f' is a functional on $\mathcal{D}$, in the middle f is a functional on $\mathcal{D}$, while in the integral on the right f is a pointwise-valued function. From the definition of the pointwise-valued function f, integrating by parts or invoking the fundamental theorem of calculus, this is

$$-A \cdot \int_a^b \varphi'(x)\, dx - B \cdot \int_b^c \varphi'(x)\, dx \;=\; -A \cdot (\varphi(b) - \varphi(a)) - B(\varphi(c) - \varphi(b)) \;=\; (A \cdot \delta_a + (B - A) \cdot \delta_b - B \cdot \delta_c)(\varphi)$$

as claimed. ///

[06.5] Show that the principal value functional $u(\varphi) = P.V. \int_{\mathbb{R}} \frac{\varphi(x)}{x}\, dx$ satisfies $x \cdot u = 1$.

Discussion: For $\varphi \in \mathcal{D}$,

$$u(\varphi) \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \frac{x \cdot \varphi(x)}{x}\, dx \;=\; \lim_{\varepsilon \to 0^+} \int_{|x| \geq \varepsilon} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} \varphi(x)\, dx \;=\; \int_{\mathbb{R}} 1 \cdot \varphi(x)\, dx \;=\; 1(\varphi)$$

since φ is continuous at 0. Thus, $x \cdot u = 1$. ///

[06.6] Compute the Fourier transform of the sign function

$$\text{sgn}(x) \;=\; \begin{cases} 1 & (\text{for } x > 0) \\ \\ -1 & (\text{for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx}\mathrm{sgn} = 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi ix$, this implies that $(2\pi i)x \cdot \widehat{\mathrm{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x \cdot \widehat{\mathrm{sgn}} = 1$.

Discussion: From the hint, $x \cdot (\pi i\,\widehat{\mathrm{sgn}}) = 1$. Also, the principal-value functional u from the previous example satisfies $x \cdot u = 1$. Thus,

$$x \cdot (u - \pi i\,\widehat{\mathrm{sgn}}) \;=\; 0$$

By another earlier example, this implies that $u - \pi i\,\widehat{\mathrm{sgn}}$ is a multiple of δ. In fact, the multiple is 0, because δ is *even*, while u, sgn, and thus $\widehat{\mathrm{sgn}}$, are all *odd*.[1] That is, $\widehat{\mathrm{sgn}} \;=\; \frac{1}{\pi i}u$. ///

[0.1] Remark: In particular, it is *not quite* that $\widehat{\mathrm{sgn}}(\xi) = 1/\pi i\xi$. Indeed, $1/\xi$ is *not* locally integrable, so does not directly describe a distribution. This example shows that, yes, $\xi \cdot \widehat{\mathrm{sgn}} = 1/\pi i$, but apparently we cannot just *divide* (pointwise values). Indeed, we have proven that the principal-value integral is the Fourier transform (up to constants), and it is not quite just an integral.

[06.7] Show that $x\delta' = \delta$ on $\mathbb{R}$. Similarly, on $\mathbb{R}^n$, show that $x_i\delta = 0$.

Discussion: These are direct computations, using the characterizations of multiplication and of derivative by duality. For the first assertion, for $\varphi\mathscr{S}$,

$$(x\delta')(\varphi) \;=\; \delta'(x\cdot\varphi) \;=\; -\delta((x\varphi)') \;=\; -\delta(\varphi + x\varphi') \;=\; -\delta(\varphi) + 0\cdot\varphi'(0) \;=\; -\delta(\varphi)$$

as claimed. On $\mathbb{R}^n$, for $\varphi \in \mathscr{S}$,

$$(x_i\delta)(\varphi) \;=\; \delta(x_i\varphi) \;=\; 0\cdot\varphi(0) \;=\; 0$$

as claimed. ///

[06.8] On $\mathbb{R}^n$, show that $\Delta\delta = 2n\cdot\delta$.

Discussion: Another direction computation, using the duality characterization: for $\varphi \in \mathscr{S}$,

$$(r^2\Delta\delta)(\varphi) \;=\; (\Delta\delta)(r^2\varphi) \;=\;=\; (-1)^2\delta(\Delta(r^2\varphi))$$

Compute

$$\Delta(r^2\varphi) \;=\; \sum_i \frac{\partial^2}{\partial x_i^2}(r^2\varphi) \;=\; \sum_i \frac{\partial}{\partial x_i}(2x_i\varphi + r^2\frac{\partial\varphi}{\partial x_i})$$

$$= \sum_i 2\varphi + 2x_i\frac{\partial\varphi}{\partial x_i} + r^2\frac{\partial^2\varphi}{\partial x_i^2} \;=\; 2n\varphi + \sum_i 2x_i\frac{\partial\varphi}{\partial x_i} + nr^2\Delta\varphi$$

Applying δ to this gives

$$2n\varphi(0) + \sum_i 2\cdot 0\cdot\frac{\partial\varphi}{\partial x_i}(0) + n\cdot 0\cdot(\Delta\varphi)(0) \;=\; 2n\varphi(0) \;=\; 2n\delta(\varphi)$$

as claimed. ///

[06.9] On $\mathbb{R}^2$, compute the Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2+y^2)}$ for $n = 0, 1, 2, \ldots$. (*Hint:* Re-express things, including Fourier transform, in terms of $z = x + iy$ and $\overline{z} = x - iy$, $w = u + iv$, and $\overline{w} = u - iv$.)

[1] This notion of parity can be defined for distributions from the obvious notion for functions $(\theta\cdot f)(x) = f(-x)$, and then $(\theta\cdot v)(f) = v(\theta\cdot f)$ for distributions v.

Discussion: Using z and w, the functions are $z^n e^{-\pi z\overline{z}}$ and $\overline{z}^n e^{-\pi z\overline{z}}$, and Fourier transform is

$$\int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, z^n e^{-\pi z\overline{z}}\, dx\, dy \;=\; \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, \frac{1}{(-\pi)^n}\Big(\frac{\partial}{\partial\overline{z}}\Big)^n e^{-\pi z\overline{z}}\, dx\, dy$$

Imagining that we can integrate by parts, this is

$$(-1)^n \frac{1}{(-\pi)^n}\int_{\mathbb{R}^2}\Big(\frac{\partial}{\partial\overline{z}}\Big)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; \frac{1}{\pi^n}\int_{\mathbb{R}^2}(-\pi i w)^n e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy$$

$$=\; (-i)^n\, w^n \int_{\mathbb{R}^2} e^{-\pi i(z\overline{w}+\overline{z}w)}\, e^{-\pi z\overline{z}}\, dx\, dy \;=\; i^{-n}\, w^n\, e^{-\pi(w\overline{w})}$$

since we know the Fourier transform of a Gaussian. A similar computation with roles of $z, \overline{z}$ reversed accomplishes the other computation. That is, $(x \pm iy)^n e^{-\pi(x^2+y^2)}$ is an eigenfunction for Fourier transform, with eigenvalue $i^{-|n|}$. ///
