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Examples discussion 07

Paul Garrett garrett@math.umn.edu http://www.math.umn.edu/~garrett/

[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-07.pdf]

[07.1] Compute the Fourier transform of $|x|$ on $\mathbb{R}$. (Hint: its second derivative is 2δ.)

(Beware! There was a computational error in the discussion from last year!)

Discussion: (There are several lines of computation which succeed.) Let $u(x) = |x|$, or, rather, the distribution given by integrating against $|x|$. This is certainly a tempered distribution, so it has a Fourier transform, even if it is not (integration against) a pointwise-valued function. Its first derivative is (integration against) $\operatorname{sgn} x$, which has no pointwise value at 0, but that doesn't matter. The derivative of this is 0 away from 0, and, more importantly, gives a jump of 2 at 0, so $u'' = 2\delta$.

Taking Fourier transform, $(2\pi i\xi)^2\,\widehat{u} = 2$, since $\widehat{\delta} = 1$. We are reasonably tempted to divide through and say that $\widehat{u}(\xi) = -1/2\pi^2|\xi|^2$. However, this cannot be literally correct, since $1/|x|^2$ is not locally integrable, so this description of the distribution $\widehat{u}$ is inadequate. Also, attempting a naive *principal value* version fails because there's no cancellation.

But we might be reminded of the earlier example that the principal value distribution $v(f) = \lim_{\varepsilon\to 0^+}\int_{|x|\geq\varepsilon}\frac{f(x)}{x}\,dx$ appears as a Fourier transform:

$$\widehat{\operatorname{sgn}} \;=\; \frac{1}{\pi i}\,v$$

In that case, we similarly saw that $\xi\,\widehat{\operatorname{sgn}} = 1/\pi i$, but we could not simply *divide*, due to problems at 0. Rather, since also $\xi\cdot v = 1$, $\xi\cdot(\widehat{\operatorname{sgn}} - v/\pi i) = 0$. Thus, $\widehat{\operatorname{sgn}} - v/\pi i$ is supported at 0, so is a linear combination of δ and its derivatives. Being annihilated by multiplication by ξ, it must be a constant multiple of δ itself. But $\widehat{\operatorname{sgn}} - v/\pi i$ is *odd*, and δ is *even*, so it must be that $\widehat{\operatorname{sgn}} - v/\pi i = 0$.

Still, we might imagine that, since $(1/\xi)' = -1/|\xi|^2$ away from 0, $\widehat{u}$ may be the *derivative* of the principal value functional (up to a constant multiple). Taking the derivative of both sides of $\xi\cdot v = 1$, we have $v + \xi v' = 0$, so $\xi(-v') = v$. Multiplying again by ξ,

$$\xi^2\cdot(-v') \;=\; \xi\cdot v \;=\; 1$$

Thus,

$$\xi^2\cdot(\widehat{u} - v'/2\pi^2) \;=\; \frac{-1}{2\pi^2} + \frac{1}{2\pi^2} \;=\; 0$$

Thus, $\widehat{u} - v'/2\pi^2$ is supported at 0, so is a (finite) linear combination of δ and its derivatives. Being annihilated by multiplication by ξ^2, it is necessarily just a linear combination of δ and δ'. Since $\widehat{u}, v', \delta$ are even, but δ' is odd, the linear combination can only involve δ. So we know that

$$\widehat{u} \;=\; \frac{v'}{2\pi^2} + c\cdot\delta \qquad\qquad \text{(for some constant } c\text{)}$$

That is, up to the to-be-determined multiple of δ, $\widehat{u}$ is essentially the derivative of the principal-value functional. Unlike the previous, simpler example, we need to evaluate the constant. To do so, it's handy to use a Schwartz function that is its own Fourier transform, such as the Gaussian $g(x) = e^{-\pi x^2}$. Then

$$c\cdot 1 \;=\; c\cdot\delta(g) \;=\; \widehat{u}(g) - \frac{v'}{2\pi^2}(g) \;=\; u(\widehat{g}) + \frac{v}{2\pi^2}(g') \;=\; \int_{\mathbb{R}} |x|\, e^{-\pi x^2}\; dx + \frac{1}{2\pi^2}\lim_{\varepsilon}\int_{|x|\geq\varepsilon} \frac{-2\pi x\, e^{-\pi x^2}}{x}\; dx$$

$$= \frac{1}{\pi} - \frac{1}{\pi} \;=\; 0$$

Thus, $\widehat{|x|} = v'/2\pi^2$, with v the principal-value functional above. ///

[0.1] Remark: This can also be understood as an example of Hadamard's *finie partie* (finite part), as well as in terms of Riesz's explanation of Hadamard's idea in terms of meromorphic continuation of a family of distributions. All of these viewpoints are useful. (A previous year's computation of the constant was apparently incorrect!)

[07.2] (*Trace Theorem* $\mathbb{T}^2 \to \mathbb{T}^1$) For $f \in H^s(\mathbb{T}^2)$ with $s > \frac{1}{2}$, show that $f\Big|_{\mathbb{T}\times\{0\}} \in H^{s-\frac{1}{2}}(\mathbb{T})$.

Discussion: Let's recall the more general case of this done in class: let $m < n$ and $\mathbb{T}^m \to \mathbb{T}^n$ by mapping $(x_1,\ldots,x_m) \to (x_1,\ldots,x_m,0,\ldots,0)$.

[0.2] Claim: For $s > \frac{n-m}{2}$, for $f \in H^s(\mathbb{T}^n)$, the restriction $f|_{\mathbb{T}^m}$ is in $H^{s-\frac{n-m}{2}-\varepsilon}(\mathbb{T}^m)$ for every $\varepsilon > 0$.

Proof: Fix $\varepsilon > 0$, and let $h = \frac{n-m}{2} + \varepsilon$. Denote elements of $\mathbb{Z}^n$ by (k,ℓ) with $k \in \mathbb{T}^m$ and $\ell \in \mathbb{T}^{n-m}$. Also, it suffices to consider f having a *finite* Fourier series, since these are dense in every H^s, so we do not have to worry about convergence, only *comparison of norms*. Then

$$\Big|f\Big|_{\mathbb{T}^m}\Big|^2_{H^{s-h}} \;=\; \sum_{k\in\mathbb{Z}^m}\Big|\sum_{\ell\in\mathbb{Z}^{n-m}} \widehat{f}(k,\ell)\Big|^2\cdot(1+|k|^2)^{s-h}$$

By Cauchy-Schwarz-Bunyakowsky,

$$\Big|\sum_{\ell\in\mathbb{Z}^{n-m}}\widehat{f}(k,\ell)\Big| \;\leq\; \sum_{\ell\in\mathbb{Z}^{n-m}} 1\cdot|\widehat{f}(k,\ell)| \;=\; \sum_{\ell\in\mathbb{Z}^{n-m}} \frac{1}{(1+|\ell|^2)^{h/2}}\cdot|\widehat{f}(k,\ell)|\cdot(1+|\ell|^2)^{h/2}$$

$$\leq\; \Big(\sum_{\ell\in\mathbb{Z}^{n-m}}\frac{1}{(1+|\ell|^2)^h}\Big)^{\frac{1}{2}}\cdot\Big(\sum_{\ell\in\mathbb{Z}^{n-m}}|\widehat{f}(k,\ell)|^2\cdot(1+|\ell|^2)^h\Big)^{\frac{1}{2}}$$

Since $h > \frac{n}{2}$, the first sum has finite value C_h. Then

$$\Big|f\Big|_{\mathbb{T}^m}\Big|_{H^{s-h}} \;\leq\; C_h\cdot\sum_{k\in\mathbb{Z}^m}\Big(\sum_{\ell\in\mathbb{Z}^{n-m}}|\widehat{f}(k,\ell)|^2\cdot(1+|\ell|^2)^h\Big)\cdot(1+|k|^2)^{s-h}$$

Since $s-h \geq 0$ and $h \geq 0$, for all $a,b \geq 0$,

$$(1+a)^h\cdot(1+b)^{s-h} \;\leq\; (1+a+b)^h\cdot(1+a+b)^{s-h} \;=\; (1+a+b)^s$$

Thus,

$$\Big|f\Big|_{\mathbb{T}^m}\Big|^2_{H^{s-h}} \;\leq\; C_h\cdot\sum_{k\in\mathbb{Z}^m}\sum_{\ell\in\mathbb{Z}^{n-m}}|\widehat{f}(k,\ell)|^2\cdot(1+|k|^2+|\ell|^2)^s \;=\; C_h\cdot|f|^2_{H^s}$$

This comparison of norms on finite Fourier series extends by continuity to give the same comparison for all elements of H^s. ///

[07.3] Let $\psi_\xi(x) = e^{2\pi i\xi\cdot x}$. Tell in what useful sense $\int_{\mathbb{R}^n} 1\cdot\psi_\xi\; d\xi$ converges.

Discussion: It is easy to imagine that the integral should converge in some genuine sense, and express δ, since $\widehat{\delta} = 1$, and Fourier inversion holds for tempered distributions. This is the $\mathbb{R}^n$ analogue of the $\mathbb{T}^n$ situation with Fourier series having coefficients all 1, the Fourier expansion of the Dirac comb (a periodic version of Dirac δ). But for tempered distributions there is no reason to expect pointwise convergence either of Fourier transform or inversion, so from this viewpoint that integral can only refer to an extension-by-continuity of Fourier inversion. We can do somewhat better.

First, since $\widehat{\delta} = 1$ is locally integrable, and $\int_{\mathbb{R}^n} |1|^2 \cdot (1+|\xi|^2)^s \, d\xi < \infty$ for $s < -\frac{n}{2}$, we find that $\delta \in H^s(\mathbb{R}^n)$ for such s. We might aim to show that the integral in question converges (in some sense) to δ in H^s for such s.

Just as Fourier series need not be interpreted as converging *numerically* to *pointwise*-valued functions, these Fourier integrals need not be interpreted as converging numerically to pointwise-valued functions. One point is that infinite sums or integrals over infinite-measure sets should be construed as *limits* of (for example) finite sums or finite integrals.

We claim that this integral converges in the Sobolev space $H^s(\mathbb{R}^n)$ for every $s < -n/2$, and converges there to δ. For example, the *truncated* integrals

$$u_N \;=\; \int_{\sup_i |\xi_i| \le N} 1 \cdot \psi_\xi \, d\xi$$

are absolutely convergent pointwise, so can be taken literally. Taking advantage of the box-truncations (rather than other shapes that do not easily allow separation of variables),

$$u_N(x) \;=\; \prod_{i=1}^n \frac{\sin 2\pi N x_i}{\pi x_i}$$

We should expect that $u_N \to \delta$ in $H^s(\mathbb{R}^n)$. Indeed, by Fourier inversion, $\widehat{u_B}$ is the characteristic function χ_{B_N} of the box $B_N = \{x : \sup_i |x_i| \le N\}$. Then

$$|\delta - u_N|^2_{H^s} \;=\; \int_{\mathbb{R}^n} |1 - \chi_{B_N}|^2 \cdot (1+|\xi|^2|^s \, d\xi \;\le\; \int_{|x| \ge N} (1+|\xi|^2)^s \, d\xi \;\longrightarrow\; 0$$

for $s < -\frac{n}{2}$. ///

[07.4] Show that there exists $f \in C^o(\mathbb{R}^n)$ and $0 \le k \in \mathbb{Z}$ such that $(1-\Delta)^k f = \delta$.

Discussion: The key idea is that solving the equation $(1-\Delta)f = g$ gives $(1+4\pi^2|\xi|^2)\widehat{f}(\xi) = \widehat{g}(\xi)$, and then $\widehat{f}(\xi) = \widehat{g}(\xi)/(1+4\pi^2|\xi|^2)$. Thus, $(1-\Delta)^k f = g$ gives $\widehat{f}(\xi) = \widehat{g}(\xi)/(1+4\pi^2|\xi|^2)^k$. This puts f in a better Sobolev space than g was in, shifting the index by $2k$. If the index is shifted to H^s with $s > \frac{n}{2} + \ell$, then by Sobolev inequalities/imbedding, actually $f \in C^\ell$.

Again, $\delta \in H^s(\mathbb{R}^n)$ for any $s < -\frac{n}{2}$, since $\widehat{\delta} = 1$ and $\int |1|^2 \cdot (1+|\xi|^2)^s \, d\xi < \infty$. Let $F(\xi) = 1/(1+4\pi^2|\xi|^2)^k$ with k large enough so that for some $\varepsilon > 0$

$$\int_{\mathbb{R}^n} \frac{1}{(1+4\pi^2|\xi|^2)^{2k}} \cdot (1+|\xi|^2)^{\frac{n}{2}+\varepsilon} d\xi \;<\; \infty$$

With $f = \widehat{F}$, $f \in H^{\frac{n}{2}+\varepsilon}$, and $(1-\Delta)^k f = \delta$. By Sobolev imbedding, $f \in C^o$, as desired. ///

[07.5] Show that the characteristic function of an interval is in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{1}{2}}(\mathbb{R})$.

Discussion: By direct computation,

$$\widehat{\chi}_{[a,b]}(\xi) \;=\; \frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{-2\pi i\xi}$$

For $\varepsilon > 0$,

$$\int_{\mathbb{R}} \Big|\frac{e^{-2\pi i b\xi} - e^{-2\pi i a\xi}}{-2\pi i\xi}\Big|^2 \cdot (1+\xi^2)^s \, d\xi \;\ll\; \int_{\mathbb{R}} \frac{1}{(1+|\xi|)^2} \cdot (1+\xi^2)^s \, d\xi \;<\; \infty$$

for $2 - 2s > 1$, which is $s < \frac{1}{2}$. ///

[07.6] (Corrected!) Show that $f(x) = e^{-|x|}$ is in $H^{\frac{3}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{3}{2}}(\mathbb{R})$.

Discussion: The correct indexes for the Sobolev spaces are easily discovered by doing a simple computation: basic calculus gives

$$\widehat{f}(\xi) \;=\; \frac{2}{1+4\pi^2\xi^2}$$

and then, with implicit constant that doesn't matter,

$$|f|^2_{H^s} \;=\; \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \cdot (1+\xi^2)^s \, d\xi \;\ll\; \int_{\mathbb{R}} (1+\xi^2)^{s-2} \, d\xi$$

This if finite for $s - 2 < -\frac{1}{2}$, which is $s < \frac{3}{2}$. On the other hand, with implicit constants that don't matter,

$$|f|^2_{H^s} \;=\; \int_{\mathbb{R}} |\widehat{f}(\xi)|^2 \cdot (1+\xi^2)^s \, d\xi \;\gg\; \int_{\mathbb{R}} (1+\xi^2)^{s-2} \, d\xi$$

which diverges for $s - 2 = -\frac{1}{2}$. ///

[07.7] Recall the argument that $\delta \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ for every ε, but is *not* in $H^{-\frac{n}{2}}(\mathbb{R}^n)$.

Discussion: This is a an important cliche:

$$\int_{\mathbb{R}^n} |\widehat{\delta}|^2 \cdot (1+|\xi|^2)^s \, d\xi \;=\; \int_{\mathbb{R}^n} 1 \cdot (1+|\xi|^2)^s \, d\xi$$

which is finite for $s < -\frac{n}{2}$, but not for $s = -\frac{n}{2}$. ///

[07.8] Let u be a distribution on $\mathbb{R}$. Show that $\delta * u = u$ and $\delta' * u = u'$.

Discussion: Use the definition/characterization

$$(u * f)(x) \;=\; u(T_{-x} f^\vee)$$

where $f^\vee(y) = f(-y)$, for $u \in \mathcal{E}^*$ and $f \in \mathcal{E}$. Then

$$(\delta * f)(x) \;=\; \delta(T_{-x} f^\vee) \;=\; (T_{-x}(y \to f(-y)))|_{y=0} \;=\; (y \to f(-y+x))|_{y=0} \;=\; f(x)$$

Similarly,

$$(\delta' * f)(x) \;=\; \delta'(T_{-x} f^\vee) \;=\; -(\frac{d}{dy}(y \to T_{-x} f(-y))|_{y=0} \;=\; -(\frac{d}{dy}(y \to f(-y+x)))|_{y=0}$$

$$=\; (y \to f'(-y+x))|_{y=0} \;=\; f'(x)$$

as claimed. ///

Another reasonable approach, is to use Fourier transforms: apparently

$$\widehat{\delta * u} \;=\; \widehat{\delta} \cdot \widehat{u} \;=\; 1 \cdot \widehat{u} \;=\; \widehat{u}$$

and by Fourier inversion it would seem that $\delta * u = u$. Indeed, if $\widehat{u}$ has *pointwise values* this argument is correct. Thus, for $u \in \mathcal{E}^*$, since by Sobolev $H^\infty \subset \mathcal{E}$ and then $\mathcal{E}^* \subset (H^\infty)^* = H^{-\infty}$, we know that $\widehat{u}$ *does* have pointwise values, justifying the argument. ///

Also, if $\widehat{u}$ is a compactly-supported distribution, multiplication of it by any $f \in \mathcal{E}$ is defined, not pointwise, but by duality, by $(f \cdot \widehat{u})(\varphi) = \widehat{u}(f \cdot \varphi)$ for $\varphi \in \mathcal{E}$. ///

[07.9] For compactly supported distributions u, v on $\mathbb{R}$, show that $(u * v)' = u' * v = u * v'$.

Discussion: First, there is an argument from the definitions: for $f \in \mathcal{E}$, using *associativity*,

$$((u*v)' * f)(x) \;=\; (u*v)'(T_{-x} f^\vee) \;=\; -(u*v)(\frac{d}{dy}(y \to f(-y+x))) \;=\; -u * (v * (\frac{d}{dy}(y \to f(-y+x))))$$

Then, going back, this is

$$-u * (v(T_{-x}\frac{d}{dy}(y \to f(-y)))) \;=\; -u * (v(\frac{d}{dy}T_{-x}(y \to f(-y)))) \;=\; u * (v'(T_{-x}(y \to f(-y))))$$

$$=\; u * (x \to v' * f(x)) \;=\; ((u * v') * f)(x)$$

as claimed. ///

Another approach uses the idea that $\delta' * u = u' = u * \delta'$ for $u \in \mathcal{E}^*$, together with *associativity*. Namely,

$$u * v' \;=\; u * (v * \delta') \;=\; (u * v) * \delta' \;=\; (u * v)'$$

and

$$u' * v \;=\; (\delta' * u) * v \;=\; \delta' * (u * v) \;=\; (u * v)'$$

This might motivate us to think again why δ' behaves this way on $\mathcal{E}^*$, not only on $\mathcal{E}$. ///

While we're here, let's explicitly prove associativity of $*$ for $u, v, w \in \mathcal{E}^*$: for every $f \in \mathcal{E}$, using the associativity $(u * v) * f = u * (v * f)$ that essentially defines $u * v$,

$$(u * (v * w)) * f \;=\; u * ((v * w) * f) \;=\; u * (v * (w * f)) \;=\; (u * v) * (w * f) \;=\; ((u * v) * w) * f$$

as desired. ///

[07.10] Let H be the Heaviside step function (with $H' = \delta$). Let 1 denote the identically-one function. Verify that $(1 * \delta') * H = 0$, while $1 * (\delta' * H) = 1$, so *associativity fails*:

$$(1 * \delta') * H \;=\; 0 \;\neq\; 1 \;=\; 1 * (\delta' * H)$$

(This is not a pathology, because there is no purposeful definition of convolution involving two or more general not-compactly-supported distributions.)

Discussion: *[... iou ...]*

[07.11] (*) Show that the functional on $f \in \mathcal{D}(\mathbb{R}^2)$ given by integrating around the unit circle

$$u(f) \;=\; \int_0^{2\pi} f(\cos\theta,\, \sin\theta)\, d\theta$$

is in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^2)$ [**Terrible typo: I had $\mathbb{R}^1$ in the original...**] for every $\varepsilon > 0$.

Discussion: The (correct) idea is that restriction to a smooth, nicely imbedded submanifold reduces the Sobolev index by half the codimension divided by 2, plus epsilon. To carry this out precisely, we'd need to choose some change-of-coordinates to flatten out the round circle, to reduce to the cases of $\mathbb{T}^m \subset \mathbb{T}^n$ or $\mathbb{R}^m \subset \mathbb{R}^n$. ///
