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Examples discussion 08

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[08.1] Recall the proof of the spectral theorem for self-adjoint operators on a finite-dimensional complex vector space V with hermitian inner product. Recall the proof of a spectral theorem for *two* self-adjoint operators S, T on V under the assumption that $ST = TS$.

Discussion: First, we prove that *any* linear map T on a finite-dimensional complex vector space V has a non-zero eigenvector. Let $\varphi : \mathbb{C}[x] \to \mathrm{End}_{\mathbb{C}}(V)$ be the $\mathbb{C}$-linear ring homomorphism defined by $\varphi(x) = T$. Since $\mathbb{C}[x]$ is infinite-dimensional over $\mathbb{C}$, and $\mathrm{End}_{\mathbb{C}}(V)$ is finite-dimensional (by the finite-dimensionality of V), the kernel of φ is non-zero. It is an ideal in $\mathbb{C}[x]$, and since $\mathbb{C}[x]$ is a principal ideal domain (because $\mathbb{C}$ is a field), $\ker\varphi = \langle P\rangle$ for some monic polynomial P. This is the *minimal polynomial* of T. Since $\mathbb{C}$ is algebraically closed, we can factor $P(x) = (x - \alpha_1)\ldots(x - \alpha_n)$ with $\alpha_i \in \mathbb{C}$. Since this is the minimal polynomial of T, letting $Q(x) = (x - \alpha_2)\ldots(x - \alpha_n)$, there is $v_o \in V$ such that $Q(T)(v_o) \neq 0$. But

$$0 \;=\; P(T)(v_o) \;=\; (T - \alpha_1)\Big(Q(T)(v_o)\Big)$$

so $Q(T)(v_o)$ is a non-zero α_1-eigenvector of T. ///

Next, with such v_o, for $T = T^*$ for a hermitian inner product $\langle,\rangle$ on V, we claim that $v_o^\perp$ is a T-stable subspace of V, and that the restriction of T to $v_o^\perp$ is still hermitian. Indeed, for $w \in v_o^\perp$,

$$\langle Tw, v_o\rangle \;=\; \langle w, Tv_o\rangle \;=\; \langle w, \lambda v_o\rangle \;=\; \overline{\lambda}\cdot\langle w, v_o\rangle \;=\; \overline{\lambda}\cdot 0 \;=\; 0$$

so $v_o^\perp$ is T stable. Then the hermitian-ness of the restriction of T to that space is immediate.

By induction on the dimensions of vector spaces with hermitian operators, the restriction of T to $v_o^\perp$ has an orthonormal basis of eigenvectors. Adjoining v_o to this orthonormal basis gives an orthonormal basis of eigenvectors for V.

Given another self-adjoint operator S on V, first observe that S stabilizes the λ-eigenspace V_λ of T: for v in that eigenspace, by associativity,

$$T(Sv) \;=\; (TS)(v) \;=\; (ST)(v) \;=\; S(Tv) \;=\; S(\lambda v) \;=\; \lambda\cdot Sv$$

The restriction of S to V_λ is still hermitian, so V_λ has an orthonormal basis of S-eigenvectors. These are also T-eigenvectors. Since V is the orthogonal direct sum of the T-eigenspaces, this gives the desired orthonormal basis of simultaneous S-and-T-eigenvectors. ///

[08.2] Let $K(x, y) = |x - y|$, and let

$$Tf(x) \;=\; \int_a^b K(x,y)\, f(y)\, dy \qquad\qquad (\text{for } f \in L^2[a,b])$$

Find some eigenvalues/eigenfunctions for the operator T. (*Hint*: consider $\frac{d^2}{dx^2}(Tf)$ and use the fundamental theorem of calculus.)

Discussion: Take $\lambda \neq 0$. First, a bootstrapping procedure shows that a λ-eigenfunction f is at least C^1, as follows. From $\lambda\cdot f(x) = \int_a^b |x - y|\, f(y)\, dy$, with $0 \le x < x' \le 1$,

$$|f(x) - f(x')| \;\le\; \frac{1}{|\lambda|}\int_0^1 \Big||x' - y| - |x - y|\Big|\cdot|f(y)\, dy$$

There are three cases: $0 \le y \le x$, $x < y < x'$, and $y \ge x'$. In the first and third, $||x'-y| - |x-y|| = |x'-x|$. In the second, we have uniform estimate

$$\Big||x'-y| - |x-y|\Big| \;\le\; |x'-y| + |x-y| \;=\; \le\; 2|x'-x|$$

Then Cauchy-Schwarz-Bunyakowsky gives

$$|f(x) - f(x')| \;\le\; \frac{1}{|\lambda|} \cdot |f|_{L^2} \cdot 2|x'-x|$$

and f is continuous. Then, invoking the fundamental theorem of calculus, the eigenfunction property expressses f as a finite linear combination of C^1 functions:

$$\lambda \cdot f(x) \;=\; x\int_0^x f - \int_0^x y \cdot f(y)\, dy + \int_x^1 y \cdot f(y)\, dy - x\int_x^1 f$$

so f itself is C^1. Repeating, we find that f is C^2, justifying taking a second derivative of both sides of the equation (in a classical sense rather than distributional): first derivative is

$$\lambda \cdot f'(x) \;=\; \int_0^x f + xf(x) - xf(x) - xf(x) - \int_x^1 f + xf(x) \;=\; \int_0^x f - \int_x^1 f$$

and the second derivative is

$$\lambda \cdot f''(x) \;=\; f(x) + f(x) \;=\; 2f(x)$$

(For $\lambda \neq 0$) this gives the constant-coefficient equation $f'' = \frac{2}{\lambda} f$, which has solutions consisting of linear combinations of $x \to e^{x \cdot \sqrt{2/\lambda}}$.

With $f(x) = e^{cx}$, the integral transform can be evaluated by breaking the integral into pieces and integrating by parts:

$$Tf(x) \;=\; \int_0^1 |x-y| \cdot e^{cy}\, dy \;=\; \int_0^x (x-y) \cdot e^{cy}\, dy - \int_x^1 (x-y) \cdot e^{cy}\, dy$$

$$= \Big[(x-y) \cdot \frac{e^{cy}}{c}\Big]_{y=0}^x - \int_0^x (-1) \cdot \frac{e^{cy}}{c}\, dy - \Big[(x-y) \cdot \frac{e^{cy}}{c}\Big]_{y=x}^1 + \int_x^1 (-1) \cdot \frac{e^{cy}}{c}\, dy$$

$$= \Big(0 - x\frac{1}{c}\Big) + \Big[\frac{e^{cy}}{c^2}\Big]_{y=0}^x - \Big((x-1)\frac{e^c}{c} - 0\Big) - \Big[\frac{e^{cy}}{c^2}\Big]_{y=x}^1$$

$$= -\frac{x}{c} + \Big(\frac{e^{cx}}{c^2} - \frac{1}{c^2}\Big) - (x-1)\frac{e^c}{c} - \Big(\frac{e^c}{c^2} - \frac{e^{cx}}{c^2}\Big) \;=\; 2\frac{e^{cx}}{c^2} - x\Big(\frac{1}{c} + \frac{e^c}{c}\Big) + \Big(-\frac{1}{c^2} + \frac{e^c}{c} - \frac{e^c}{c^2}\Big)$$

Such f cannot be an eigenfunction unless the linear terms vanish identically. We must examine the extent to which linear combinations $f(x) = Ae^{cx} + Be^{-cx}$ may cause the extra terms to cancel, under conditions on $c = \sqrt{2/\lambda}$: with such f,

$$T(Ae^{cx} + Be^{-cx}) \;=\; 2\frac{Ae^{cx} + B^{-cx}}{c^2} - x\Big(\frac{A}{c} + \frac{B}{-c} + \frac{Ae^c}{c} + \frac{Be^{-c}}{-c}\Big) + \Big(-\frac{A}{c^2} - \frac{B}{(-c)^2} + \frac{Ae^c}{c} + \frac{Be^{-c}}{-c} - \frac{Ae^c}{c^2} - \frac{Be^{-c}}{(-c)^2}\Big)$$

For the linear term to vanish identically, the coefficient of x and the constant coefficient must be 0. This gives a homogeneous system of two equations in the two unknowns A, B:

$$\begin{cases} \Big(\frac{1}{c} + \frac{e^c}{c}\Big) \cdot A & - & \Big(\frac{1}{c} + \frac{e^{-c}}{c}\Big) \cdot B & = & 0 \\ \Big(\frac{-1}{c^2} + \frac{e^c}{c} - \frac{e^c}{c^2}\Big) \cdot A & - & \Big(\frac{1}{c^2} + \frac{e^{-c}}{c} + \frac{e^{-c}}{c^2}\Big) \cdot B & = & 0 \end{cases}$$

This has a non-trivial solution if and only if the determinant is zero, multiplying through by c^3, this condition is

$$0 \;=\; -(1+e^c)\cdot(1+ce^{-c}+e^{-c}) + (-1+ce^c-e^c)\cdot(1+e^{-c})$$
$$=\; -(1+ce^{-c}+e^{-c}+e^c+c+1) + (-1+ce^c-e^c-e^{-c}+c-1)$$
$$=\; -1-ce^{-c}-e^{-c}-e^c-c-1-1+ce^c-e^c-e^{-c}+c-1$$
$$=\; -4-ce^{-c}-2e^{-c}-2e^c+ce^c \;=\; -2(e^{c/2}+e^{-c/2})^2 + c(e^c-e^{-c})$$
$$=\; (e^{c/2}+e^{-c/2})\Big(c(e^{c/2}-e^{-c/2}) - 2(e^{c/2}+e^{-c/2})\Big)$$

For $e^{c/2}+e^{-c/2}=0$, the first factor is 0, so $c/2 \in \frac{\pi i}{2}+\pi i\mathbb{Z}$ gives an eigenfunction. That is, $c\in \pi i + 2\pi i\mathbb{Z}$. Without worrying about possible zeros of the other factor of the determinant, taking $A=1$, since the first equation in the system vanishes identically, we look at the second, obtaining

$$B \;=\; \frac{-1+ce^c-e^c}{1+ce^{-c}+e^{-c}} \;=\; \frac{ce^c - e^{c/2}(e^{c/2}+e^{-c/2})}{ce^{-c}+e^{-c/2}(e^{c/2}+e^{-c/2})} \;=\; \frac{ce^c}{ce^{-c}} \;=\; e^{2c} \;=\; 1 \qquad (\text{since } c\in\pi i+2\pi i\mathbb{Z})$$

Thus, for $c\in\pi i+2\pi i\mathbb{Z}$,

$$e^{cx}+e^{-cx}$$

is an eigenfunction. That is, *at least* $\cos\pi x$, $\cos 3\pi x$, $\cos 5pix, \ldots$ are eigenfunctions. ///

[0.1] Remark: As might be suspected, there are more eigenvalues and eigenvectors, corresponding to zeros of the second factor in the determinant, but the eigenvalues λ and corresponding parameters $c=\sqrt{2/\lambda}$ are not as elementarily expressible as in the previous case. Indeed, with $c=ib$ and $b\in\mathbb{R}$, with $e^{c/2}+e^{-c/2}\neq 0$, vanishing of the second factor is

$$0 \;=\; ib(e^{ib/2}-e^{-ib/2}) - 2(e^{ib/2}+e^{-ib/2}) \;=\; b(e^{ib/2}+e^{-ib/2})\cdot\Big(-\frac{e^{ib/2}-e^{-ib/2}}{i(e^{ib/2}+e^{-ib/2})} - \frac{2}{b}\Big)$$
$$=\; b(e^{ib/2}+e^{-ib/2})\cdot\Big(-\tan\frac{b}{2}-\frac{2}{b}\Big)$$

Since $\tan\frac{b}{2}$ is periodic and goes from $-\infty$ to $+\infty$ in each period, it intersects the curve $(b,-2/b)$ in each period. In fact, since both are monotone, they intersect exactly once in each period. That is, there is (at least) another batch of eigenvalues at least as numerous (asymptotically) as the previous.

This can be anticipated on general principles, if we observe that any function f expressible as a superposition of functions $x\to|x-y|$ on $[0,1]$ satisfies the *boundary conditions* $f'(0)+f'(1)=0$ and $\int_0^1 f = f(0)+f(1)$, since $x\to|x-y|$ has those properties. That is, rather than having *no* conditions on a function on the circle $\mathbb{R}/\mathbb{Z}$, giving eigenfunctions $e^{2\pi inx}$, with eigenvalues $(2\pi in)^2$ with multiplicity *two* (for $n\neq 0$), the eigenvalues get pushed farther from 0 by the implicit boundary conditions.

Variant: To see what happens without the complications entailed by restricting to a finite interval, we can consider

$$Tf(x) \;=\; \int_{\mathbb{R}} |x-y|\cdot f(y)\,dy \qquad (\text{for } f\in C_c^o(\mathbb{R}))$$

Certainly *something* is required for convergence, since $y\to|x-y|$ is not in $L^2(\mathbb{R})$. Again, this is

$$Tf(x) \;=\; \int_{y\le x}(x-y)\cdot f(y)\,dy - \int_{y\le x}(x-y)\cdot f(y)\,dy$$

$$=\; x\int_{y\le x} f(y)\,dy - \int_{y\le x} yf(y)\,dy - x\int_{y\ge x} f(y)\,dy + \int_{y\ge x} yf(y)\,dy$$

The continuity of f assures that both integrals are continuously differentiable, by the fundamental theorem of calculus. Thus,

$$(Tf)'(x) \;=\; \int_{y\le x} f(y)\,dy + xf(x) - xf(x) - \int_{y\ge x} f(y)\,dy + xf(x) - xf(x) \;=\; \int_{y\le x} f(y)\,dy - \int_{y\ge x} f(y)\,dy$$

and the second derivative is $2f(x)$. Although there are no eigenfunctions, this shows that, given $f \in C^o_c(\mathbb{R})$, we can solve the equation $u'' = f$ by $u(x) = \int_{\mathbb{R}} |x-y| \cdot f(y)\,dy$. ///

[08.3] Let $K(x,y) \in L^2([a,b]\times[a,b])$, and attempt to define a map $T : L^2[a,b] \to L^2[a,b]$ by

$$Tf(x) \;=\; \int_a^b K(x,y)\,f(y)\,dy$$

Show that Tf is well-defined a.e. as a pointwise-valued function. Show that T really does map L^2 to itself by showing that

$$|Tf|_{L^2[a,b]} \;\le\; |K|_{L^2([a,b]\times[a,b])} \cdot |f|_{L^2[a,b]}$$

(One would say that $K(,)$ is a *Schwartz kernel* for the map T. Yes, this use is in conflict with the use of *kernel* of a map to refer to things that map to 0.) In the previous situation, show that the Hilbert-space adjoint T^* of T has Schwartz kernel $\overline{K(y,x)}$.

Discussion: By Fubini-Tonelli, $y \to K(x,y)$ is measurable for almost all x, so $Tf(x)$ is defined almost everywhere (assuming convergence of the integral). By Cauchy-Schwarz-Bunyakowsky, and Fubini-Tonelli as needed,

$$\int_a^b |Tf(x)|^2\,dx \;=\; \int_a^b \Big|\int_a^b K(x,y)\,f(y)\,dy\Big|^2\,dx \;\le\; \int_a^b \int_a^b |K(x,y)|^2\,dy \cdot \int_a^b |f(y')|^2\,dy'\,dx$$

$$=\; |f|^2_{L^2} \cdot \int_a^b \int_a^b |K(x,y)|^2\,dx\,dy \;=\; |f|^2_{L^2[a,b]} \cdot |K|^2_{L^2([a,b]\times[a,b])} \;<\; +\infty$$

Thus, T is *bounded*, so is a *continuous* linear map of $L^2[a,b]$ to itself. ///

[08.4] Prove that the *Volterra operator* $Vf(x) = \displaystyle\int_0^x f(t)\,dt$ on $C^o[0,1]$ or on $L^2[0,1]$ has no (not-identically-zero) eigenvalues/eigenvectors.

Discussion: It suffices to consider $f \in L^2[0,1]$, since $C^o[0,1] \subset L^2[0,1]$. First consider $\lambda \ne 0$. The initial step is a sort of *bootstrapping* process to see that any eigenfunction $f \in L^2[0,1]$ would have to be in $C^1[0,1]$ (and, in fact, in $C^\infty[0,1]$). For $f \in L^2[0,1]$ and $0 \le x < y \le 1$, by Cauchy-Schwarz-Bunyakowsky,

$$|f(y) - f(x)| \;=\; \Big|\frac{1}{\lambda}\int_x^y f(t)\,dt\Big| \;\le\; \frac{1}{|\lambda|}\Big(\int_x^y |f|^2\Big)^{\frac12} \cdot \Big(\int_x^y 1\Big)^{\frac12} \;\le\; \frac{1}{|\lambda|}|f|_{L^2} \cdot |y-x|^{\frac12}$$

giving continuity. For continuous f such that $\lambda \cdot f(x) = \int_0^x f(t)\,dt$, since the integrand is continuous, the integral is C^1 as a function of x, by the fundamental theorem of calculus. Differentiating both sides of the equation, $\lambda \cdot f'(x) = f(x)$ for all x. Also, the integral is 0 at $x = 0$, so $f(0) = 0$. We claim that the constant-coefficient differential equation $\lambda \cdot f' - f = 0$ with condition $f(0) = 0$ has only the zero solution. Indeed, all the solutions are of the form $f(x) = c \cdot e^{x/\lambda}$ for some constant c. (We can *prove* this widely-believed fact via the Mean Value Theorem: write a solution f as $f(x) = e^x \cdot g(x)$ for $g(x) = f(x)/e^x$. Then the differential equation becomes $\lambda(e^{x/\lambda} \cdot g)' - (e^x \cdot g) = 0$, which simplifies to $g' = 0$. The Mean Value Theorem assures us that g is constant.) Thus, $f(0) = 0$ implies $c = 0$, and f must be identically 0.

For $\lambda = 0$, the equation $0 = \int_0^x f(t)\,dt$ holds identically in x, so $\int_x^y f = 0$ for all $0 \le x < y \le 1$. That is, such $f \in L^2[0,1]$ is orthogonal to all characteristic functions of intervals. It is plausible that this implies that $f = 0$

(in the $L^2[0,1]$ sense). We can take advantage of the fact that we know that $1, \ldots \sin 2\pi nx, \cos 2\pi nx, \ldots$ for $n = 1, 2, 3, \ldots$ is an orthonormal basis for $L^2[0,1]$: if we can show that the L^2-closure of the span of characteristic functions of intervals contains all functions $\sin 2\pi nx, \cos 2\pi nx$, then f would be orthogonal to all these, hence 0. Indeed, the usual discussion of Riemann integrals of continuous functions is more than enough to show that continuous functions can be approximated arbitrarily well by linear combinations of characteristic functions of intervals. ///

[08.5] Determine the spectrum of the *left-shift* $L : (c_1, c_2, \ldots) \to (c_2, \ldots)$ on ℓ^2, and of the *right-shift* $R : (c_1, c_2, \ldots) \to (0, c_1, c_2, \ldots)$ on ℓ^2. Show that these are mutual *adjoints*.

Discussion: The adjoint characterization is $\langle Tv, w\rangle = \langle v, T^* w\rangle$. That means that, for $(w_1, w_2, \ldots)$ in ℓ^2, we want

$$\langle (z_1, z_2, \ldots),\ R^*(w_1, w_2, \ldots)\rangle \ = \ \langle R(z_1, z_2, \ldots),\ (w_1, w_2, \ldots)\rangle \ = \ \langle (0, z_1, z_2, \ldots),\ (w_1, w_2, \ldots)\rangle$$

$$= \ z_1 w_2 + z_2 w_3 + z_3 w_4 + \ldots \ = \ \langle (z_1, z_2, \ldots),\ (w_2, w_3, \ldots)\rangle \ = \ \langle (z_1, z_2, \ldots),\ L(w_1, w_2, \ldots)\rangle$$

Thus, we see that $R^* = L$. ///

To determine the spectrum of R, first identify the discrete spectrum (eigenvalues) of R. An eigenvalue assumption

$$(0, z_1, z_2, \ldots) \ = \ R(z_1, z_2, \ldots) \ = \ \lambda \cdot (z_1, z_2, \ldots) \ = \ (\lambda z_1, \lambda z_2, \ldots)$$

gives $\lambda z_1 = 0$, $\lambda z_2 = z_1$, $\lambda z_3 = z_2$, ... If $\lambda \neq 0$, then $\lambda z_1 = 0$ implies $z_1 = 0$, and, by induction, $z_n = 0$ for all indices n. For $\lambda = 0$, $\lambda z_{n+1} = z_n$ implies $z_n = 0$ for all n. Thus, the right shift has no discrete spectrum.

For the left shift, the eigenfunction condition gives $\lambda z_1 = z_2$, $\lambda z_2 = z_3$, ... Let i be the smallest index such that $z_i \neq 0$. Then $z_n = \lambda^{n-i} \cdot z_i$ for $n \geq i$. For $|\lambda| < 1$, this produces an eigenvector in ℓ^2. For $|\lambda| \geq 1$ this is impossible except for the 0-vector.

[1.2] Remark: Note that even though ℓ^2 is a *separable* Hilbert space, meaning that it has a countable orthonormal basis, the left shift has *uncountably* many distinct eigenvalues and eigenvectors.

To determine the whole spectrum of the left-shift, we can use general properties: the spectrum $\sigma(T)$ of a (continous linear) operator T is a compact set, and is inside the ball of radius $|T|_{\text{op}}$. Both left and right shift do not make any vector larger, so their operators norms are both ≤ 1. Thus, since the left shift L has eigenvalues $\{\lambda \in \mathbb{C} : |\lambda| <\}$, $\sigma(L)$ contains at least the closure of this open ball. On the other hand, it cannot be larger than that (by the operator norm estimate). Thus, $\sigma(L)$ is *exactly* the closed unit ball.

To understand the whole spectrum of the right-shift R, we can prove a more broadly useful result:

[1.3] Claim: For $\lambda \in \sigma_d(T)$ and $\overline{\lambda} \notin \sigma_d(T^*)$, $\overline{\lambda}$ is in the residual spectrum of T^*.

Proof: Let $v_o \neq 0$ be a λ-eigenvector for T. We claim that the image of $T^* - \overline{\lambda}$ is orthogonal to v_o: for any v,

$$\langle (T^* - \overline{\lambda})v, v_o\rangle \ = \ \langle v, (T - \lambda)v_o\rangle \ = \ \langle v, 0\rangle = 0$$

Since $T^* - \overline{\lambda}$ is injective by assumption, $\overline{\lambda}$ is in the residual spectrum of T^*. ///

Thus, since $\{\lambda \in \mathbb{C} : |\lambda| < 1\} = \sigma_d(L)$ and $\sigma_d(R) = \sigma_d(L^*) = \phi$, the residual spectrum of R contains at least the open unit disk. Since it is closed, and bounded by the operator norm 1, the spectrum of R is the closed unit disk. ///

[08.6] The Volterra operator $Vf(x) = \int_0^x f(y)\, dy$ on $L^2[0,1]$ has kernel

$$K(x,y) \ = \ \begin{cases} 1 & (\text{for } 0 \leq y \leq x \leq 1) \\ \\ 0 & (\text{for } 0 \leq x < y \leq 1) \end{cases}$$

Determine the (Schwartz) kernel for $T = V \circ V^*$. Find some eigenfunctions for T. (Recall that V has no eigenfunctions!) (*Hint:* apply d/dx to the equation $Tf = \lambda \cdot f$ and presume that the differentiation passes inside the integral.)

Discussion: The kernel $K^*(x, y)$ for an adjoint V^* is always obtained by the procedure $\overline{K(y, x)}$. Here, the kernel is real-valued. For two operators S, T with respective kernels $K_S(x, y)$ and $K_T(x, y)$, using Fubini-Tonelli as necessary,

$$(S \circ T)f(x) \;=\; (S(Tf))(x) \;=\; \int_0^1 K_S(x,t)\, Tf(t)\, dt \;=\; \int_0^1 K_S(x,t)\Big(\int_0^1 K_T(t,y)\, f(y)\, dy\Big)\, dt$$

$$= \int_0^1 \Big(\int_0^1 K_S(x,t)\, K_T(t,y)\, dt\Big)\, f(y)\, dy$$

so the kernel of the composite is

$$K_{S\circ T}(x,y) \;=\; \int_0^1 K_S(x,t)\, K_T(t,y)\, dt$$

Thus, the kernel $L(x, y)$ for $V \circ V^*$ is

$$K_{V\circ V^*}(x,y) \;=\; \int_0^1 K_V(x,t)\, K_V(y,t)\, dt \;=\; \int_0^1 \begin{cases} 1 & (\text{for } 0 \le t \le x \le 1) \\ 0 & (\text{for } 0 \le x < t \le 1) \end{cases} \cdot \begin{cases} 1 & (\text{for } 0 \le t \le y \le 1) \\ 0 & (\text{for } 0 \le y < t \le 1) \end{cases} dt$$

$$= \int_{t \le x,\ t \le y} 1\, dt \;=\; \min(x,y)$$

$$\int_0^1 \min(x,y)\, f(y)\, dy \;=\; \int_0^x y\, f(y)\, dy + \int_x^1 x\, f(y)\, dy \;=\; \int_0^x y\, f(y)\, dy + x \int_x^1 f(y)\, dy$$

For $f \in L^2$, Cauchy-Schwarz-Bunyakowsky shows that the latter expression side is *continuous* as a function of x. Thus, an eigenfunction equation $\lambda \cdot f = (V \circ V^*)f$ for $f \in L^2$ and $\lambda \neq 0$ implies that f is *continuous*. Then from

$$\lambda \cdot f(x) \;=\; \int_0^x y\, f(y)\, dy + x \int_x^1 f(y)\, dy$$

the fundamental theorem of calculus implies that $f \in C^1$. By induction on k, $f \in C^k$ for all k, so f is *smooth*. Differentiating the latter expression,

$$\lambda \cdot f'(x) \;=\; x \cdot f(x) + \int_x^1 f(y)\, dy - x \cdot f(x) \;=\; \int_x^1 f(y)\, dy$$

Differentiating again, $\lambda \cdot f'' = -f$. Thus, $f(x) = Ae^{cx} + Be^{-cx}$ for some constants A, B, with $c = \sqrt{-\lambda}$. But this is only a *necessary* condition for an eigenfunction, not *sufficient*.

One way to determine allowable λ is to directly compute the integral

$$\int_0^1 \min(x,y) \cdot (Ae^{cy} + Be^{-cy})\, dy$$

and examine the condition that this be equal to $-1/c^2 \cdot (Ae^{cx} + Be^{-cx})$. This would involve two integrations by parts. Equivalently, but somewhat more lightly, for each fixed $x \in [0, 1]$,

$$F(y) \;=\; \begin{cases} 0 & (\text{for } y < 0) \\ y & (\text{for } 0 \le y < x) \\ x & (\text{for } x \le y < 1) \\ 0 & (\text{for } y \ge 1) \end{cases}$$

Integrating by parts twice *distributionally* gives

$$\int_0^1 \min(x,y)\cdot(Ae^{cy}+Be^{-cy})\,dy \;=\; \int_{\mathbb{R}} F(y)\cdot(Ae^{cy}+Be^{-cy})\,dy \;=\; \int_{\mathbb{R}} F''(y)\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy$$

$$= \int_{\mathbb{R}} \Big(\Big(\begin{cases} 0 & (\text{for } y<0) \\ 1 & (\text{for } 0\le y<x) \\ 0 & (\text{for } y\ge x)\end{cases}\Big) - x\cdot\delta_1\Big)'\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy$$

$$= \int_{\mathbb{R}} \Big(\delta_0-\delta_x-x\cdot\delta_1'\Big)\cdot\frac{Ae^{cy}+Be^{-cy}}{c^2}\,dy \;=\; \frac{A+B}{c^2}-\frac{Ae^{cx}+Be^{-cx}}{c^2}-x\cdot\frac{Ace^{c}-Bce^{-c}}{c^2}$$

For this to be $-(Ae^{cx}+Be^{-cx})/c^2$, it is necessary and sufficient that the two extra terms vanish, that is,

$$\frac{A+B}{c^2}-x\cdot\frac{Ace^{c}-Bce^{-c}}{c^2} \;=\; 0$$

for almost all $x\in[0,1]$. This is a linear function in x, so $A+B=0$ and then $e^c+e^{-c}=0$, which is $e^{2c}=-1$. That is, $c\in\pi i+2\pi i\mathbb{Z}$ with corresponding eigenfunction $e^{cx}-e^{-cx}$. ///

[08.7] *(Approximate eigenvectors and continuous spectrum, Weyl's criterion)* Let $T:V\to V$ be a self-adjoint linear operator on a Hilbert space V. For $\lambda\in\mathbb{C}$, a sequence $\{v_n\}$ of vectors (normalized so that all their lengths are 1 or at least bounded away from 0) such that $(T-\lambda)v_n\to 0$ as $n\to+\infty$ is an *approximate eigenvector* for λ. Show that for λ *not* an eigenvalue for T, λ has an approximate eigenvector if and only if λ is in the spectrum of T.

Discussion: In fact, this criterion is *not* reliable for detecting some types of *residual* spectrum, which is why we impose self-adjointness to exclude it. [1] We give an example at the end of the discussion. Certainly if λ is an eigenvector, with non-zero eigenvalue v, the constant sequence $v,v,v,\ldots$ fits the requirement.

For general spectrum, let $S=T-\lambda$. For $v_1,v_2,\ldots$ with $|v_n|=1$ and $Sv_n\to 0$, any alleged (continuous [2]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 \;=\; S^{-1}(\lim_n Sv_n) \;=\; \lim_n S^{-1}Sv_n \;=\; \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T-\lambda$ implies that $T-\lambda$ is not invertible.

Conversely, for $S=T-\lambda$ not invertible, but λ not an eigenvector, then S is *injective* but not *surjective*. We do need a further assumption: suppose that the image of S is *not closed*. [3]

[1.4] Theorem: For λ *not* an eigenvalue for T, and for $(T-\lambda)V$ *not closed*, λ is in the spectrum of T if and only if λ has an approximate eigenvector.

[1.5] Remark: This criterion is *not* uniformly reliable for detecting *residual* spectrum, which is why we must impose a further condition. [4] For example, we have seen that, for $T:V\to V$ a *norma* linear operator, for

[1] Recall that *residual* spectrum of T is λ such that $T-\lambda$ is *injective*, but does *not* have dense image.

[2] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S, necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta>0$ such that $|Sv|\ge\delta\cdot|v|$ for all v. This exactly asserts the boundedness of S^{-1}, so S^{-1} is *continuous*.

[3] The image is not closed, for example, when T (hence S) has no residual spectrum, which is the case when T (hence S) is *normal*, or *self-adjoint*.

[4] Recall that *residual* spectrum of T is λ such that $T-\lambda$ is *injective*, but does *not* have dense image.

λ in the spectrum but not an eigenvalue, $(T-\lambda)V$ is *dense* in V but is not all of V. Thus, the hypothesis of the theorem is met for normal T. We give an example of failure to detect residual spectrum after the proof.

Proof: Certainly if λ is an eigenvector, with non-zero eigenvalue v, the constant sequence $v, v, v, \ldots$ fits the requirement.

For general spectrum, let $S = T - \lambda$. For $v_1, v_2, \ldots$ with $|v_n| = 1$ and $Sv_n \to 0$, any alleged (continuous [5]) S^{-1} would give, interchanging S^{-1} and the limit by continuity,

$$0 \;=\; S^{-1}(\lim_n Sv_n) \;=\; \lim_n S^{-1}Sv_n \;=\; \lim_n v_n$$

contradiction. Thus, existence of an approximate eigenvector for $T-\lambda$ implies that $T-\lambda$ is not invertible.

Conversely, for $S = T - \lambda$ not invertible, but λ not an eigenvector, then S is *injective* but not *surjective*. We further assume that the image of S is *not closed*. [6] In that case, there is v_o (with $|v_o| = 1$) not in the image of S, and $v_1, v_2, \ldots$ such that $Sv_1, Sv_2, \ldots \to v_o$. If $\{v_n\}$ were a Cauchy sequence, then it would have a limit, and by continuity of S

$$v_o \;=\; \lim_n Sv_n \;=\; S(\lim_n v_n)$$

and v_o would be in the image of S, contradicting our assumption. Thus, $\{v_n\}$ is *not* Cauchy. In particular, we can replace $\{v_n\}$ by a subsequence such that there is $\delta > 0$ such that $|v_m - v_n| \ge \delta$ for all $m \ne n$. Then $w_n = v_n - v_{n+1}$ forms an approximate 0-eigenvector, since their lengths are bounded away from 0, and

$$Sw_n \;=\; S(v_n - v_{n+1}) \;=\; Sv_n - Sv_{n+1} \;\longrightarrow\; v_o - v_o \;=\; 0$$

as desired. ///

As noted, the case that λ is not an eigenvector, $T-\lambda$ is not surjective, and/but the image of $S = T - \lambda$ is *closed*, can only occur for non-normal T. For example, $T : \ell^2 \to \ell^2$ by

$$T(c_1, c_2, \ldots) \;=\; (c_1, 0, c_2, 0, c_3, 0, \ldots)$$

is injective, not surjective, and has closed image. It is not invertible, but there is no approximate eigenvector for 0, so the criterion fails in this (non-normal) example. ///

[08.8] Show that the multiplication operator $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ by $Tf(x) = f(x)\cdot\sin x$ has empty discrete spectrum. Show that it is self-adjoint. Show that T has continuous spectrum the interval $[-1, 1]$. (We know that self-adjoint (or merely *normal*) operators have only point spectum and continuous spectrum, that is, no left-over *residual* spectrum.)

Discussion: This operator is self-adjoint, since $\sin x$ is real-valued:

$$\langle Tf, g\rangle \;=\; \int_{\mathbb{R}} f(x)\cdot\sin x\cdot\overline{g(x)}\,dx \;=\; \int_{\mathbb{R}} f(x)\cdot\overline{g(x)\cdot\sin x}\,dx \;=\; \langle f, Tg\rangle$$

For a function f and fixed $\lambda \in \mathbb{C}$ such that $f(x)\cdot\sin x = \lambda\cdot f(x)$ for almost all x, for x such that $f(x) \ne 0$, necessarily $\sin x = \lambda$. Since $\sin x$ assumes any particular value at most countably many times, $f = 0$ almost everywhere. Thus, there are no eigenvalues.

[5] Recall that when there is an everywhere-defined, linear inverse S^{-1} to S, necessarily S is a continuous bijection, and by the *open mapping theorem* S is *open*. That is, there is $\delta > 0$ such that $|Sv| \ge \delta\cdot|v|$ for all v. This exactly asserts the boundedness of S^{-1}, so S^{-1} is *continuous*.

[6] The image is not closed, for example, when T (hence S) has no residual spectrum, which is the case when T (hence S) is *normal*, or *self-adjoint*.

Since T is self-adjoint, it is normal, so there is no residual spectrum. Thus, Weyl's criterion via approximate eigenvectors suffices to determine the remainder of the spectrum, which will be *continuous*. Given a value $\lambda \in [-1,1]$, let $x_o \in \mathbb{R}$ be such that $\sin x_o = \lambda$. We claim that an approximate eigenvector for λ can be formed by functions concentrated ever-more-closely at x_o, such as

$$v_n(x) \;=\; \begin{cases} \sqrt{n} & (\text{for } |x-x_o| \le \frac{1}{2n}) \\ \\ 0 & (\text{otherwise}) \end{cases}$$

By design, $|v_n| = 1$. Since $\sin x$ is continuous, given $\varepsilon > 0$ there is $\delta > 0$ such that $|\sin x - \sin x_o| < \varepsilon$ for $|x - x_o| < \delta$. For n large enough so that $1/2n < \delta$,

$$|Tv_n - \lambda v_n|^2_{L^2} \;=\; |v_n \cdot \sin x - \lambda \cdot v_n|^2_{L^2} \;=\; \int_{x_o-\frac{1}{2n}}^{x_o+\frac{1}{2n}} n \cdot |\sin x - \sin x_o|^2\, dx \;<\; \int_{x_o-\frac{1}{2n}}^{x_o+\frac{1}{2n}} n \cdot \varepsilon^2\, dx \;=\; \varepsilon^2$$

Thus, $Tv_n - \lambda v_n \to 0$, and $\{v_n\}$ is an approximate identity for λ, so every $\lambda \in [-1,1]$ is in the continuous spectrum. ///

[08.9] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0,1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0,1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

Discussion: Since the set $\{|r_1|, |r_2|, \ldots\}$ is bounded by 1, the operator norm of T is at most 1, so it is bounded, hence continuous. Since the r_n are all *real*, the operator is self-adjoint:

$$\langle T(a_1, a_2, \ldots), (b_1, b_2, \ldots)\rangle \;=\; \langle (r_1a_1, r_2a_2, \ldots), (b_1, b_2, \ldots)\rangle \;=\; \sum_n r_n a_n \cdot \overline{b_n}$$

$$= \sum_n a_n \cdot \overline{r_n b_n} \;=\; \langle (a_1, a_2, \ldots), T(b_1, b_2, \ldots)\rangle$$

When $\lambda \cdot (c_1, c_2, \ldots) = T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots)$, necessarily $\lambda \cdot c_n = r_n \cdot c_n$ for all n. When $c_n \neq 0$, this implies $\lambda = r_n$. Since the r_n are distinct, there can be (at most) one index n for which $c_n \neq 0$, and then $\lambda = r_n$. Conversely, every r_n is obviously an eigenvalue.

Since we know that the whole spectrum is *closed* in $\mathbb{C}$, it contains at least the closure of the rationals in $[0,1]$, namely, $[0,1]$ itself. Since T is self-adjoint, its spectrum is contained in $\mathbb{R}$. [7] Since the spectrum is bounded by $|T|_{\text{op}} = 1$, it is contained in $[-1,1]$.

To see that $\lambda \in [-1,0)$ is *not* in the spectrum, in that $(T-\lambda)(c_1, c_2, \ldots) = ((r_1-\lambda)c_1, (r_2-\lambda)c_2, \ldots)$, we have $|r_n - \lambda| \ge |\lambda| > 0$, so the inverse $(T-\lambda)^{-1}$ can be written down immediately: $(T-\lambda)^{-1}(c_1, c_2, \ldots) =$

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[7] The proof that self-adjoint operators T have spectrum inside $\mathbb{R}$ has more content than just the analogous assertion about eigenvectors. For $Tv = \lambda v$ with $v \neq 0$, of course

$$\lambda\langle v, v\rangle \;=\; \langle \lambda v, v\rangle \;=\; \langle Tv, v\rangle \;=\; \langle v, Tv\rangle \;=\; \langle v, \lambda v\rangle \;=\; \overline{\lambda}\langle v, v\rangle$$

shows that any eigenvalues are real. Since self-adjoint operators have no residual spectrum, to find the rest of the spectrum it suffices to identify *approximate eigenvectors*. Note that for self-adjoint T always $\langle Tv, v\rangle = \overline{\langle v, Tv\rangle} = \overline{\langle Tv, v\rangle}$, so $\langle Tv, v\rangle$ is *real*. Then for $(T-\lambda)v_n \to 0$, certainly $\langle (T-\lambda)v_n, v_n\rangle \to 0$, so the *imaginary parts* go to 0. These are

$$\text{Im}\,\langle (T-\lambda)v_n, v_n\rangle \;=\; \text{Im}\,\langle Tv_n, v_n\rangle + \text{Im}\,(\lambda \cdot \langle v_n, v_n\rangle) \;=\; 0 + \text{Im}\,(\lambda) \cdot \langle v_n, v_n\rangle$$

Since $|v_n|$ are bounded away from 0, there can be an approximate identity only for $\lambda \in \mathbb{R}$. ///

$((r_1 - \lambda)^{-1}c_1, (r_2 - \lambda)^{-1}c_2, \ldots)$ and there is a uniform upper bound $|(r_n - \lambda)^{-1}| \le |\lambda|^{-1}$. [8] Finally, given irrational $\lambda \in [0,1]$, let $r_{n_1}, r_{n_2}, \ldots$ be rationals such that $r_{n_i} \to \lambda$. With standard basis $\{e_n\}$ for ℓ^2, we claim that $\{e_{n_i}\}$ is an approximate eigenvector for λ: given $\varepsilon > 0$, let N be sufficiently large so that $|r_{n_i} - \lambda| < \varepsilon$ for $i \ge N$. For $n_i \ge N$,

$$|(T-\lambda)e_{n_i}| \;=\; |(r_{n_i} - \lambda)e_{n_i}| \;=\; |r_{n_i} - \lambda|_{\mathbb{C}} \cdot |e_{n_i}|_{\ell^2} \;=\; |r_{n_i} - \lambda|_{\mathbb{C}} \;<\; \varepsilon$$

Thus, indeed, $(T - \lambda)e_{n_i} \to 0$, and the e_{n_i} give an approximate identity for λ, so λ is in the spectrum.
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[8] For such a simple operator, a similar device shows that $\lambda \notin \mathbb{R}$ is not in the spectrum.