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Examples discussion 09

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[09.1] Show that the translation action $T_x f(y) = f(y+x)$ on the Banach space $C^o_{bdd}(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ is *not* continuous. That is, $\mathbb{R} \times C^o_{bdd}(\mathbb{R}) \to C^o_{bdd}(\mathbb{R})$ by $x \times f \to T_x f$ is *not* continuous. In particular, find a particular $f \in C^o_{bdd}(\mathbb{R})$ with $|f|_{C^o} = 1$ such that, there is a sequence $\delta_n \to 0$ of non-zero numbers δ_n such that $|T_{\delta_n} f - f|_{C^o} = 1$.

Discussion: The point is that on a non-compact topological space there may exist continuous, bounded, but *not uniformly continuous* functions, such as $f(x) = \sin(x^2)$. Let $x_n = n \cdot \sqrt{2\pi}$ and let $\delta_n > 0$ be a sequence of small positive reals going to 0 such that $(x_n + \delta_n)^2 = x_n^2 + \frac{\pi}{2}$. Then $\sin(x_n^2) = 0$, while $\sin((x_n+\delta_n)^2) = 1$, so the sup norm of $\sin(x^2) - \sin((x+\delta_n)^2)$ is 1. ///

[9.1] Remark: Nevertheless, the translation action *is* continuous on $C^o_o(\mathbb{R})$, which we see as follows. Given $f \in C^o_o(\mathbb{R})$, for given $\varepsilon > 0$, by a previous example there is $g \in C^o_c(\mathbb{R})$ such that $\sup_{x\in\mathbb{R}} |g(x) - f(x)| < \varepsilon$. Since g is compactly supported, it is *uniformly* compact, so there is $\delta > 0$ such that $|x-y| < \delta$ implies $|g(x) - g(y)| < \varepsilon$. Then for $|h| < \delta$,

$$\sup_{x\in\mathbb{R}} |f(x+h) - f(x)| \;\le\; \sup_{x\in\mathbb{R}} |f(x+h) - g(x+h) - (f(x) - g(x))| + \sup_{x\in\mathbb{R}} |g(x+h) - g(x)|$$

$$\le\; \sup_{x\in\mathbb{R}} |f(x+h) - g(x+h)| + \sup_{x\in\mathbb{R}} |f(x) - g(x)| + \sup_{x\in\mathbb{R}} |g(x+h) - g(x)| \;<\; \varepsilon + \varepsilon + \varepsilon$$

This is half the desired continuity, in contrast to the problem with $C^o_{bdd}(\mathbb{R})$. Similarly, the translation action $\mathbb{R} \times C^o_o(\mathbb{R}) \to C^o_o(\mathbb{R})$ is jointly continuous in both arguments.

[09.2] Let $r_1, r_2, r_3, \ldots$ be an enumeration of the rational numbers inside the interval $[0,1]$. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$. Show that T is a continuous/bounded linear operator, is self-adjoint, has eigenvalues exactly the $r_1, r_2, \ldots$, and continuous spectrum the whole interval $[0,1]$ (with rationals removed, if one insists on disjointness of discrete and continuous spectrum).

Discussion: Since the set $\{|r_1|, |r_2|, \ldots\}$ is bounded by 1, the operator norm of T is at most 1, so it is bounded, hence continuous. Since the r_n are all *real*, the operator is self-adjoint:

$$\langle T(a_1, a_2, \ldots), (b_1, b_2, \ldots)\rangle \;=\; \langle (r_1 a_1, r_2 a_2, \ldots), (b_1, b_2, \ldots)\rangle \;=\; \sum_n r_n a_n \cdot \overline{b_n}$$

$$= \sum_n a_n \cdot \overline{r_n b_n} \;=\; \langle (a_1, a_2, \ldots), T(b_1, b_2, \ldots)\rangle$$

When $\lambda \cdot (c_1, c_2, \ldots) = T(c_1, c_2, \ldots) = (r_1 c_1, r_2 c_2, \ldots)$, necessarily $\lambda \cdot c_n = r_n \cdot c_n$ for all n. When $c_n \ne 0$, this implies $\lambda = r_n$. Since the r_n are distinct, there can be (at most) one index n for which $c_n \ne 0$, and then $\lambda = r_n$. Conversely, every r_n is obviously an eigenvalue.

Since we know that the whole spectrum is *closed* in $\mathbb{C}$, it contains at least the closure of the rationals in $[0,1]$, namely, $[0,1]$ itself. Since T is self-adjoint, its spectrum is contained in $\mathbb{R}$. [1] Since the spectrum is bounded by $|T|_{\rm op} = 1$, it is contained in $[-1,1]$.

[1] The proof that self-adjoint operators T have spectrum inside $\mathbb{R}$ has more content than just the analogous assertion about eigenvectors. For $Tv = \lambda v$ with $v \ne 0$, of course

$$\lambda\langle v,v\rangle \;=\; \langle \lambda v, v\rangle \;=\; \langle Tv, v\rangle \;=\; \langle v, Tv\rangle \;=\; \langle v, \lambda v\rangle \;=\; \overline{\lambda}\langle v,v\rangle$$

shows that any eigenvalues are real. Since self-adjoint operators have no residual spectrum, to find the rest of the

To see that $\lambda \in [-1,0)$ is *not* in the spectrum, in that $(T-\lambda)(c_1, c_2, \ldots) = ((r_1-\lambda)c_1, (r_2-\lambda)c_2, \ldots)$, we have $|r_n - \lambda| \geq |\lambda| > 0$, so the inverse $(T-\lambda)^{-1}$ can be written down immediately: $(T-\lambda)^{-1}(c_1, c_2, \ldots) = ((r_1-\lambda)^{-1}c_1,\ (r_2-\lambda)^{-1}c_2, \ldots)$ and there is a uniform upper bound $|(r_n-\lambda)^{-1}| \leq |\lambda|^{-1}$. [2] Finally, given irrational $\lambda \in [0,1]$, let $r_{n_1}, r_{n_2}, \ldots$ be rationals such that $r_{n_i} \to \lambda$. With standard basis $\{e_n\}$ for ℓ^2, we claim that $\{e_{n_i}\}$ is an approximate eigenvector for λ: given $\varepsilon > 0$, let N be sufficiently large so that $|r_{n_i} - \lambda| < \varepsilon$ for $i \geq N$. For $n_i \geq N$,

$$|(T-\lambda)e_{n_i}| \;=\; |(r_{n_i}-\lambda)e_{n_i}| \;=\; |r_{n_i}-\lambda|_{\mathbb{C}} \cdot |e_{n_i}|_{\ell^2} \;=\; |r_{n_i}-\lambda|_{\mathbb{C}} \;<\; \varepsilon$$

Thus, indeed, $(T-\lambda)e_{n_i} \to 0$, and the e_{n_i} give an approximate identity for λ, so λ is in the spectrum. ///

[09.3] Let $r_1, r_2, r_3, \ldots$ be a bounded sequence of complex numbers. Define $T : \ell^2 \to \ell^2$ by $T(c_1, c_2, \ldots) = (r_1c_1, r_2c_2, \ldots)$. Show that T is *compact* if and only if $r_n \to 0$.

Discussion: Let $e_1, e_2, \ldots$ be the standard (Hilbert-space) basis for ℓ^2. If the r_n do not go to 0, then there is a *subsequence* $r_{n_1}, r_{n_2}, \ldots$ bounded away from 0. Since T is compact, the images $Te_{n_i} = r_{n_i}e_{n_i}$ must have a convergent subsequence. But $|r_{n_i}e_{n_i} - r_{n_j}e_{n_j}|^2 = |r_{n_i}|^2 + |r_{n_j}|^2$ for $i \neq j$, and this is bounded away from 0, so there is no convergent subsequence, contradicting the compactness of T. Thus, in fact, $r_n \to 0$.

For the converse, perhaps the most economical approach is to observe that T is an operator-norm limit of finite-rank operators, hence compact:

$$T_n(c_1, c_2, \ldots, c_n, c_{n+1}, \ldots) \;=\; (c_1, c_2, \ldots, c_n, 0, 0, \ldots)$$

The estimate on the operator norms is

$$|T - T_n|_{\mathrm{op}} \;=\; \sup_{|v|\leq 1} |(0, \ldots, 0, r_{n+1}v_{n+1}, \ldots)| \;=\; \sup_{k \geq n} |r_k| \; To\, 0$$

Less efficiently, we can refer to definitions, and use the *total boundedness* criterion for compact closure. Given $\varepsilon > 0$, let N be large enough so that $|r_n| < \varepsilon$ for $n \geq N$. Write $v = (v_1, v_2, \ldots) \in \ell^2$ as

$$v \;=\; \underbrace{(v_1, \ldots, v_N, 0, 0, \ldots)}_{v'} + \underbrace{(\underbrace{0, \ldots, 0}_{N}, v_{N+1}, \ldots, v_{N+2}, \ldots)}_{v''}$$

Let B' be the intersection of the unit ball $B \subset \ell^2$ with the copy of $\mathbb{C}^N \subset \ell^2$ with non-zero components only at the first N places. Let B'' be the intersection of B with the subspace of ℓ^2 with 0 entries at the first N places. Certainly $B' + B'' \supset B$ and $B' \perp B''$.

By design, $|Tv''| \leq \varepsilon$ for $v'' \in B''$. Since TB' is a bounded subset of a finite-dimensional space $\mathbb{C}^N$, it has compact closure, so is totally bounded, so can be covered by finitely-many ε-balls $U_1, \ldots, U_k$. Then $TB \subset TB' + TB'' \subset (U_1 + TB'') \cup \ldots \cup (U_k + TB'')$, and every $U_i + TB''$ is contained in a 2ε-ball. Thus, TB is totally bounded, hence, has compact closure. ///

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spectrum it suffices to identify *approximate eigenvectors*. Note that for self-adjoint T always $\langle Tv, v\rangle = \overline{\langle v, Tv\rangle} = \overline{\langle Tv, v\rangle}$, so $\langle Tv, v\rangle$ is *real*. Then for $(T-\lambda)v_n \to 0$, certainly $\langle (T-\lambda)v_n, v_n\rangle \to 0$, so the *imaginary parts* go to 0. These are

$$\mathrm{Im}\,\langle (T-\lambda)v_n, v_n\rangle \;=\; \mathrm{Im}\,\langle Tv_n, v_n\rangle + \mathrm{Im}\,(\lambda \cdot \langle v_n, v_n\rangle) \;=\; 0 + \mathrm{Im}\,(\lambda) \cdot \langle v_n, v_n\rangle$$

Since $|v_n|$ are bounded away from 0, there can be an approximate identity only for $\lambda \in \mathbb{R}$. ///

[2] For such a simple operator, a similar device shows that $\lambda \notin \mathbb{R}$ is not in the spectrum.

[09.4] Let T be a compact operator $T : V \to W$ for Hilbert spaces V, W. For S a continuous/bounded operator on V, show that $T \circ S : V \to W$ is compact. For R a continuous/bounded operator on W, show that $R \circ T : V \to W$ is compact.

Discussion: For $T \circ S$, the image of the unit ball under S is contained in some ball $c \cdot B$, where B is the unit ball, because S is bounded. Since T is linear, $T(c \cdot B) = c \cdot TB$. Since TB is pre-compact, its continuous image under multiplication by c is also pre-compact. Proof: for $c = 0$, we're done. For $c > 0$, given a finite cover of TB by balls $w_i + B_\varepsilon$ where B_ε is the ball of radius $\varepsilon > 0$ centered at 0. The images $c \cdot (w_i + B_\varepsilon) = cw_i + cB_i$ cover $c \cdot TB$, and have radius $c \cdot \varepsilon$. Replacing ε by ε/c gives balls of radius ε covering $c \cdot TB$. ///

For $R \circ T$, similarly as in the previous case, given a finite cover of TB by balls $w_i + B_\varepsilon$ of radius $\varepsilon > 0$, the images $R(w_i + B_\varepsilon) = Rw_i + RB_\varepsilon$ are contained in balls $Rw_i + cB_\varepsilon$, where $c = |R|_{\text{op}}$ will suffice. ///

[09.5] Let S, T be two compact, self-adjoint operators on a Hilbert space, and $ST = TS$. Show that there is an orthonormal basis for V consisting of simultaneous eigenfunctions for S, T.

Discussion: The Hilbert space V is the closure of the orthogonal direct sum of eigenspaces V_λ for T. For $\lambda \neq 0$, V_λ is finite-dimensional, so is necessarily closed, and V_0 is the orthogonal complement of the sum of all other eigenspaces, so is closed. Since $ST = TS$, we find that S stabilizes each V_λ:

$$T(Sv) \;=\; (TS)v \;=\; (ST)v \;=\; S(Tv) \;=\; S(\lambda v) \;=\; \lambda \cdot Sv \qquad \text{(for all } v \in V_\lambda)$$

[9.2] Claim: The restriction of a compact operator to a closed subspace $W \subset V$ stabilized by it is still compact.

Proof: With B' the closed unit ball of W and B the closed unit ball of V, $TB' \subset TB$. Using the total-boundedness criterion for precompactness, given $\varepsilon > 0$, TB is covered by finitely-many ε-balls $v_i + B_\varepsilon$. Among the intersections $W \cap (v_i + B_\varepsilon)$, the non-empty ones are open balls of radius at most ε. Thus, TB' is a precompact set, and $T|_W$ is a compact operator. ///

Thus, S is a compact operator on each V_λ, so evey V_λ has an orthonormal basis of S-eigenvectors. These are also λ-eigenvectors for T, so they are simultaneous eigenvectors. ///

[09.6] Recall the proof that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals r_n,

$$C(r) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le r_n\}$$

is compact if and only if $\sum_n |r_n|^2 < \infty$.

Discussion: Use the *total boundedness* criterion. Given $\varepsilon > 0$, by convergence of $\sum_n \delta_n^2$, there is n_o large enough so that $\sum_{n \ge n_o} r_n^2 < \varepsilon^2$. The set

$$C_{n_o} \;=\; \{(z_1, z_2, \ldots, z_{n_o}) \in \mathbb{R}^{n_o} : |z_n| \le r_n\}$$

is a compact subset of $\mathbb{C}^{n_o}$, so certainly has a finite cover by open balls of radius ε. Let the centers of these balls be $w_1, \ldots, w_N$. Let $j : \mathbb{C}^{n_o} \to \ell^2$ be the inclusion $j(z_1, \ldots, z_{n_o}) = (z_1, \ldots, z_{n_o}, 0, 0, \ldots)$. Then we claim that the open balls of radius 2ε at $j(w_1), j(w_2), \ldots, j(w_N)$ cover $C(r)$. Indeed, given $z = (z_1, z_2, \ldots) \in C(r)$, write $z = j(z') + z''$ where $z' = (z_1, \ldots, z_{n_o})$ and $z'' = z - j(z') = (0, \ldots, 0, z_{n_o+1}, \ldots)$. There is at least one of the w_js within ε of z': let w_{j_o} be such. By the triangle inequality for the norm $|\cdot|_{\ell^2}$ on ℓ^2,

$$d(z, j(w_{j_o})) \;=\; |z - j(w_{j_o})|_{\ell^2} \;=\; |j(z') + z'' - j(w_{j_o})|_{\ell^2} \;\le\; |j(z') - j(w_{j_o})|_{\ell^2} + |z''|_{\ell^2}$$

$$= \; |z' - w_{j_o}|_{\mathbb{R}^{n_o}} + |z''|_{\ell^2} \;<\; \varepsilon + \varepsilon$$

Thus, $C(r)$ can be covered by finitely-many open balls of radius 2ε. ///

[09.7] First, for Schwartz φ on $\mathbb{R}^n$ and u a tempered distribution on $\mathbb{R}^n$, characterize $\varphi * u$. Show that $\widehat{\varphi * u} = \widehat{\varphi} \cdot \widehat{u}$, where the latter multiplication is that induced by duality: $(\widehat{\varphi} \cdot \widehat{u})(\psi) = \widehat{u}(\widehat{\varphi} \cdot \psi)$ for $\psi \in \mathscr{S}$. Explain why the union $H^{-\infty}$ of Sobolev spaces is inside the space of tempered distributions, and why $\widehat{u}$ has pointwise values for $u \in H^{-\infty}$.

Discussion: To anticipate a characterization of $\varphi * u$, we can examine $\varphi * u_f$ where u_f is integrate-against (for example) a locally integrable function of moderate growth, since the characterization for tempered distributions should extend (continuously...) that for distributions given by integrate-against-functions. Writing $f^\theta(x) = f(-x)$ to avoid confusion with Fourier transform notations, for $\psi \in \mathscr{S}$, invoking Fubini-Tonelli as needed to change order of integration,

$$(\varphi * u_f)(\psi) \;=\; \int_{\int} (\varphi * u_f)\, \psi \;=\; \int\!\!\int \varphi(x-y)\, f(y)\, \psi(x)\; dy\, dx \;=\; \int\!\!\int \varphi^\theta(y-x)\, f(y)\, \psi(x)\; dx\, dy$$

$$= \; \int (\varphi^\theta * \psi)(y)\, f(y)\; dx\, dy \;=\; u_f(\varphi^\theta * \psi)$$

It is important to note that $\varphi^\theta * \psi$ (with or without the θ) is still a Schwartz function. (One might reflect on the easiest way to be sure of this...) Thus, we can specify the tempered distribution $\varphi * u$ by $(\varphi * u)(\psi) = u(\varphi^\theta * \psi)$.

Since this extends the corresponding operation on distributions given by integration-against functions, but we can check once-again via this definition: for $\alpha, \beta \in \mathscr{S}$,

$$(\alpha * (\beta * u))(\psi) \;=\; (\beta * u)(\alpha^\theta * \psi) \;=\; u(\beta^\theta * (\alpha^\theta * \psi)) \;=\; u((\beta^\theta * \alpha^\theta) * \psi)$$

by associativity of convolution on Schwartz functions, which by elementary (change-of-variables) properties of θ is

$$u((\alpha * \beta)^\theta) * \psi) \;=\; ((\alpha * \beta) * u)(\psi)$$

proving the associativity.

Letting F denote Fourier transform when notationally convenient,

$$\widehat{\varphi * u}(\psi) \;=\; (\varphi * u)(\widehat{\psi}) \;=\; u(\varphi^\theta * \widehat{\psi}) \;=\; u(F(\widehat{\varphi} \cdot \psi))$$

since $\varphi\widehat{\ }\widehat{\ } = \varphi^\theta$. This is

$$\widehat{u}(\widehat{\varphi} \cdot \psi) \;=\; (\widehat{\varphi} \cdot \widehat{u})(\psi)$$

by the definition of multiplication of tempered distributions by Schwartz functions, *extending* pointwise multiplication.

Since Fourier transform maps $\mathscr{S}$ isomorphically to itself, and since $\mathscr{S}$ is certainly inside all the weighted L^2 spaces used to define the Sobolev spaces H^s, we have $\mathscr{S} \subset H^\infty$.

Since $H^\infty \subset \mathcal{E} = C^\infty$ by Sobolev imbedding, taking duals gives $\mathcal{E}^* \subset (H^\infty)^* = H^{-\infty}$. In particular, since distributions in $H^{-\infty}$ have Fourier transforms in weighted L^2 spaces, hence have pointwise values almost-everywhere, compactly-supported distributions have Fourier transforms with pointwise values almost-everywhere. (In fact, there is a Paley-Wiener theorem for compactly-supported distributions, due to L. Schwartz.)

Thus, for $u \in \mathcal{E}^*$, or even $u \in H^{-\infty}$, for $\varphi \in \mathscr{S}$, the Fourier transform $\widehat{\varphi * u} = \widehat{\varphi} \cdot \widehat{u}$ has pointwise values almost-everywhere, and thus it makes sense to assert that

$$\int_{\mathbb{R}} |\widehat{\varphi * u}|^2 \;=\; \int_{\mathbb{R}} |\widehat{\varphi} \cdot \widehat{u}|^2$$
