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Examples discussion 10

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-disc-10.pdf]

[10.1] For a bounded sequence $\lambda_1, \lambda_2, \ldots$ of complex numbers, let $T : \ell^2 \to \ell^2$ by let $T(z_1, z_2, \ldots) = (\lambda_1 z_1, \lambda_2 z_2, \ldots)$. Show that the whole spectrum of T is the (topological) closure of the set $\{\lambda_1, \lambda_2, \ldots\}$.

Discussion: Certainly the standard basis elements e_n of ℓ^2 are eigenvectors, with eigenvalues λ_n. Since the spectrum is *closed*, it suffices to prove that for λ *not* in the closure of the set of λ_n's, $(T - \lambda)^{-1}$ exists.

For $\lambda \in \mathbb{C}$ not in the closure of the set of λ_n's, there is $\delta > 0$ such that $|\lambda - \lambda_n| \ge \delta$ for all n. Then $|(\lambda - \lambda_n)^{-1}| \le \delta^{-1}$ for all n. That is,

$$(z_1, z_2, \ldots) \longrightarrow (\ldots, (\lambda - \lambda_n)^{-1} z_n, \ldots)$$

is a bounded operator on ℓ^2, visibly the two-sided inverse of T, so λ is *not* in the spectrum of T. ///

[10.2] Let $T : \ell^2_{\mathbb{Z}} \to \ell^2_{\mathbb{Z}}$ be the *two-sided right-shift* on $\ell^2_{\mathbb{Z}}$, given by $(Tz)_n = z_{n-1}$, where $z = (\ldots, z_{-1}, z_0, z_1, z_2, \ldots) \in \ell^2_{\mathbb{Z}}$. Observe that T is *normal*, in the sense that $TT^* = T^*T$, and that the adjoint T^* is the two-sided *left*-shift. Determine the eigenvalues and the whole spectrum of T.

Discussion: Indeed, $TT^* = 1_V = T^*T$, so $T^* = T^{-1}$, and T is normal. Thus, there is no residual spectrum. (Also, there is no discrete spectrum, but we do not need this fact, since we will determine the whole spectrum.) Also, $|Tv| = |v|$, so $|T|_{\text{op}} \le 1$ (in fact, $|T|_{\text{op}} = 1$). The same is true of T^*. Thus, $\sigma(T)$ and $\sigma(T^*)$ are both contained in the closed unit disk. For $\lambda \ne 0$ not in the spectrum of T,

$$(T - \lambda)^{-1} = \Big(T \circ (\lambda^{-1} - T^*) \circ \lambda\Big)^{-1} = -\lambda^{-1} \cdot (T^* - \lambda^{-1}) \circ T^*$$

That is, given that $(T - \lambda)^{-1}$ exists, *if* $(T^* - \overline{\lambda})^{-1}$ *also exists*, this formula gives it. That by itself would beg the question, so we should at least remark that this derivation is a good heuristic to obtain a proposed expression for $(T^* - \overline{\lambda})^{-1}$ in terms of $(T - \lambda)^{-1}$, and that it is easy to *check* that is correct. Thus, $(T - \lambda)^{-1}$ exists if and only if $(T^* - \lambda^{-1})^{-1}$ exists. That is, $\lambda \leftrightarrow \lambda^{-1}$ is a bijection of the (non-zero) *non-* spectrum (also called *resolvent set*) of T and T^*. Thus, there is a bijection $\sigma(T) \leftrightarrow \sigma(T^*)$ by $\lambda \leftrightarrow \lambda^{-1}$. Since the spectra are both inside the unit disk, they both must be inside the unit circle.

Now we prove that the (whole) spectrum is exactly the unit circle, by constructing approximate eigenvectors and invoking Weyl's criterion. Given $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, for $1 \le n \in \mathbb{Z}$, let

$$v_n = (\ldots, 0, 0, \underbrace{0}_{0^{th}\text{ slot}}, \underbrace{\frac{\lambda^{-1}}{\sqrt{n}}}_{1^{th}\text{ slot}}, \underbrace{\frac{\lambda^{-2}}{\sqrt{n}}}_{2^{th}\text{ slot}}, \ldots, \underbrace{\frac{\lambda^{-n}}{\sqrt{n}}}_{n^{th}\text{ slot}}, 0, 0, \ldots)$$

By design, $|v_n|_{\ell^2} = 1$.

$$(T - \lambda)v_n = (\ldots, 0, 0, \underbrace{0}_{0^{th}\text{ slot}}, \underbrace{-\frac{1}{\sqrt{n}}}_{1^{th}\text{ slot}}, \underbrace{0}_{2^{th}\text{ slot}}, \ldots, \underbrace{0}_{n^{th}\text{ slot}}, \underbrace{\frac{\lambda^{-n}}{\sqrt{n}}}_{n+1^{th}\text{ slot}}, 0, 0, \ldots)$$

This has length $\sqrt{2}/\sqrt{n} \longrightarrow 0$, so the sequence is an approximate eigenvector for λ. ///

[10.3] Let $K(,)$ be a measurable function on $\mathbb{R}^2$, with a bound B such that $\int_{\mathbb{R}} |K(x,y)|\,dx \le B$ for every y, and $\int_{\mathbb{R}} |K(x,y)|\,dy \le B$ for every x. Show that $Tf(x) = \int_{\mathbb{R}} K(x,y)\,f(y)\,dy$ gives a continuous linear map $L^p \to L^p$ for every $1 < p < \infty$, with $|Tf|_{L^p} \le B \cdot |f|_{L^p}$. (*Hint:* Hölder's inequality.)

Discussion: By Fubini-Tonelli, $y \to K(x,y)$ is measurable for almost all x, so $Tf(x)$ is defined almost everywhere (assuming convergence of the integral). By Cauchy-Schwarz-Bunyakowsky, and Fubini-Tonelli as needed,

$$\int_a^b |Tf(x)|^2\,dx \;=\; \int_a^b \Big|\int_a^b K(x,y)\,f(y)\,dy\Big|^2\,dx \;\le\; \int_a^b\int_a^b |K(x,y)|^2\,dy\cdot\int_a^b |f(y')|^2\,dy'\,dx$$

$$=\; |f|^2_{L^2}\cdot\int_a^b\int_a^b |K(x,y)|^2\,dx\,dy \;=\; |f|^2_{L^2[a,b]}\cdot|K|^2_{L^2([a,b]\times[a,b])} \;<\; +\infty$$

Thus, T is *bounded*, so is a *continuous* linear map of $L^2[a,b]$ to itself. ///

[10.4] **Simple instance of** *Young's inequality:* In the previous example, let $K(x,y) = k(x-y)$ for $k \in L^1(\mathbb{R})$, so that $Tf(x) = (k * f)(x)$. Show that $|Tf|_{L^p} \le |k|_{L^1}\cdot|f|_{L^p}$.

Discussion: One viewpoint is that this is a simple case of the previous.

Another more direct approach is an application of Minkowski's inequality:

$$|Tf|^p_{L^p} \;=\; \int_{\mathbb{R}} \Big|\int_{\mathbb{R}} k(x-y)\,f(y)\,dy\Big|^p\,dx \;\le\; \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} |k(x-y)|\,|f(y)|\,dx\Big)^p\,dy$$

by Minkowski's inequality. Since $y \to f(y)$ does not depend on the variable of integration in the inner integral, this is

$$\int_{\mathbb{R}} |f(y)|^p\Big(\int_{\mathbb{R}} |k(x-y)|\,dx\Big)^p\,dy \;=\; \int_{\mathbb{R}} |f(y)|^p\Big(\int_{\mathbb{R}} |k(x)|\,dx\Big)^p\,dy \;=\; |f|^p_{L^p}\cdot|k|^p_{L^1}$$

since the integrals separate. ///

[10.5] Write the Fourier series for the Schwartz kernel for the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$, and show that it is in $H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n})$ for every $\varepsilon > 0$.

Discussion: Let T be the identity map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)$ viewed as a map $\mathcal{D}(\mathbb{T}^n) \to \mathcal{D}(\mathbb{T}^n)^*$ via the natural imbedding $\mathcal{D} \subset \mathcal{D}^*$. Write ψ_ξ for the function $\psi_\xi(x) = e^{2\pi i\xi\cdot x}$ for $\xi \in \mathbb{Z}^n$ and $x \in (\mathbb{R}/\mathbb{Z})^n = \mathbb{T}^n$. Anticipating that there *is* a Schwartz kernel K at worst in $H^{-\infty}(\mathbb{T}^{2n}) = C^\infty(\mathbb{T}^n)$ (the latter equality by Sobolev imbedding), we can write a Fourier expansion $K = \sum_{\xi,\eta\in\mathbb{Z}^n} c_{\xi,\eta}\,\psi_\xi\otimes\psi_\eta$ with coefficients $c_{\xi,\eta}$ to be determined. [1] There is no reason to think that the Fourier series for K converges *pointwise*, and this doesn't matter. The series *does* converge in $H^{-\infty}(\mathbb{T}^{2n})$. The Schwartz kernel for $T : \mathcal{D} \to \mathcal{D}^*$ is *characterized* by

$$K(\varphi\otimes Tf) \;=\; (Tf)(\varphi) \qquad\qquad (\text{for all } \varphi\in\mathcal{D})$$

Applying this to $\varphi = \psi_\alpha$ and $f = \psi_\beta$,

$$c_{\alpha,\beta} \;=\; K(\psi_\alpha\otimes\psi_\beta) \;=\; (T\psi_\beta)(\psi_\alpha) \;=\; \int_{\mathbb{T}^n} \psi_\beta\cdot\psi_\alpha \;=\; \begin{cases} 0 & (\text{for } \beta \ne -\alpha \in \mathbb{Z}^n) \\ 1 & (\text{for } \beta = \alpha \in \mathbb{Z}^n)\end{cases}$$

The latter *necessary* condition already completely determines K: apparently $K = \sum_\alpha \psi_\alpha\otimes\psi_{-\alpha}$. However, we should give a reason why this expression really does give the identity map on $\mathcal{D}(\mathbb{T}^n)$. Certainly

$$\Big|\sum_{\alpha\in\mathbb{Z}^n} \psi_\alpha\otimes\psi_{-\alpha}\Big|^2_{H^s} \;=\; \sum_{\alpha\in\mathbb{Z}^n} |1|^2\cdot(1+|\alpha|^2)^s$$

[1] The tensor notation here is just a way to refer to the function $x, y \to \psi_\xi(x)\cdot\psi_\eta(y)$ without using arguments.

is finite if and only if $s < -\frac{n}{2}$. Thus, for every $\varepsilon > 0$, $K \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{T}^{2n}) \subset H^{-\infty}(\mathbb{T}^{2n}) = H^{\infty}(\mathbb{T}^{2n})^*$. That is, that Fourier expansion converges in a Sobolev space and does give a distribution on $\mathbb{T}^{2n}$.

Since finite linear combinations of ψ_α are *dense* in $\mathcal{D}(\mathbb{T}^n)$, and since K is continuous on $H^\infty(\mathbb{T}^n) \otimes H^\infty(\mathbb{T}^n) \subset H^\infty(\mathbb{T}^{2n})$, the earlier computation of $K(\psi_\alpha \otimes \psi_\beta)$ extends by continuity to certify that $K(f \otimes g) = \int f \cdot g$ for $f, g \in \mathcal{D}(\mathbb{T}^n)$. ///

[10.6] Write the Fourier series for the Schwartz kernel for d/dx on $\mathcal{E}(\mathbb{T})$, and tell what Sobolev space $H^s(\mathbb{T} \times \mathbb{T})$ it lies in.

Discussion: The Schwartz kernel theorem assures us that $d/dx : \mathcal{E}(\mathbb{T}) \longrightarrow \mathcal{E}(T)^*$ has a kernel $K \in \mathcal{E}(\mathbb{T}\times\mathbb{T})^*$, characterized by

$$K(\varphi \otimes f) \;=\; (\frac{d}{dx} f)(\varphi) \qquad\qquad (\text{for all } \varphi, f \in \mathcal{E}(\mathbb{T}))$$

Applying this to $\varphi = \psi_m$ and $f = \psi_n$,

$$c_{m,n} \;=\; K(\psi_m \otimes \psi_n) \;=\; (\frac{d}{dx}\psi_n)(\psi_m) \;=\; \int_{\mathbb{T}} 2\pi i n \cdot \psi_n \cdot \psi_m \;=\; \begin{cases} 0 & (\text{for } m \neq -n \in \mathbb{Z}^n) \\ 2\pi i n & (\text{for } m = -n \in \mathbb{Z}^n) \end{cases}$$

Thus,

$$K \;=\; \sum_n 2\pi i n \cdot \psi_{-n} \otimes \psi_n \;\sim\; \sum_n 2\pi i n \cdot e^{2\pi i n(y-x)} \qquad\qquad (\text{convergent in } H^{-\infty}(\mathbb{T}^2))$$

The s^{th} Sobolev norm

$$|K|^2_{H^s(\mathbb{T}^2)} \;=\; \sum_n |2\pi i n|^2 \cdot (1 + n^2 + n^2)^s$$

is finite for $s < -3/2$, so $K \in H^{-3/2-\varepsilon}(\mathbb{T}^2)$ for every $\varepsilon > 0$. ///

[10.7] **Cartan-Eilenberg adjunction:** For abelian groups A, B, C, prove that $\text{Hom}(A, \text{Hom}(B, C)) \approx \text{Hom}(A \otimes B, C)$.

Discussion: *[... iou ...]*

[10.8] $V \otimes V^* \to \text{End}_k(V)$ for finite-dimensional k-vector-spaces V, over a field k. Let V^* be its k-linear dual, $\text{Hom}_k(V, k)$. Show that the linear map $V \otimes V^* \to \text{End}_k(V)$ induced from the map $V \times V^* \to \text{End}_k(V)$ given by

$$v \times \lambda \longrightarrow \Big(w \longrightarrow \lambda(w) \cdot v\Big) \qquad\qquad (\text{with } v, w \in V \text{ and } \lambda \in V^*)$$

gives an *isomorphism* $V \otimes V^* \to \text{End}_k(V)$.

Discussion: *[... iou ...]*

[10.9] **Coordinate-independent expression for *trace*:** With finite-dimensional k-vector-space V, let $\tau : V \times_k V^* \to k$ be the natural bilinear pairing $v \times \lambda \to \lambda(v)$. Show that this map, composed with the previous, induces the *trace* map on $\text{End}_k(V)$.

Discussion: *[... iou ...]*

[10.10] Let V be a Hilbert space. Show that the *algebra* tensor product $V \otimes_{\text{alg}} V^*$ is naturally isomorphic to the *finite-rank* operators $V \to V$, with the isomorphism uniquely and completely specified by $v \otimes \lambda \to (w \to \lambda(w) \cdot v)$.

Discussion: *[... iou ...]*

[10.11] **Impossibility of extending trace from finite-rank operators to Hilbert-Schmidt operators:** Let A be the finite-rank operators on an infinite-dimensional Hilbert space V (for example, ℓ^2), identified with $V \otimes_{\text{alg}} V^*$. Let $|\cdot|_{\text{HS}}$ be the Hilbert-Schmidt norm, specified by $|v \otimes \lambda|_{\text{HS}} = |v| \cdot |\lambda|$. Show that *trace* on A does *not* extend continuously to Hilbert-Schmidt operators. Explain how this implies that trace does not extend continuously to the collection of *all* continuous linear operators (with operator norm).

Discussion: *[... iou ...]*

[10.12] For *Hermite polynomials* $H_n(x)$ defined by $H_n(x) = e^{x^2} \cdot \frac{d^n}{dx^n} e^{-x^2}$, show that the various H_n's are *orthogonal* in the weighted L^2 space $L^2(\mathbb{R}, e^{-x^2} dx)$, that is, with measure e^{-x^2} times the usual Lebesgue measure on $\mathbb{R}$.

Discussion: *[... iou ...]*
