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Examples 01

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[This document is http://www.math.umn.edu/˜garrett/m/real/notes_2017-18/real-ex-01.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 25 Sept, 2017.

[01.1] Prove that every open in $\mathbb{R}^2$ is a countable union of Cartesian products $(a,b)\times(c,d)$ of open intervals.

[01.2] Define tent functions of width w and height h centered at 0 by

$$t_{w,h}(x) \;=\; \begin{cases} 0 & (\text{for } x \le -w) \\ \frac{h}{w}\cdot(x+w) & (\text{for } -w \le x \le 0) \\ h - \frac{h}{w}\cdot x & (\text{for } 0 \le x \le w) \\ 0 & (\text{for } x \ge w) \end{cases}$$

Show that the functions $f_n(x) = t_{\frac{1}{n},n}(x - \frac{1}{n})$, a sequence of narrowing tents just to the right of 0, go to 0 *pointwise* (everywhere!), but that

$$\lim_n \int_{\mathbb{R}} f_n(x)\cdot g(x)\,dx \;=\; g(0) \qquad\qquad (\text{for all } g \in C^o(\mathbb{R}))$$

[01.3] Show that the functions $f_n(x) = t_{\frac{1}{n},n^2}(x-\frac{1}{n}) - t_{\frac{1}{n},n^2}(x+\frac{1}{n})$, whose graphs are tall tents of area n upward just to the right of 0, and tall tents downward just to the left of 0, go to 0 pointwise everywhere, but that

$$\lim_n \int_{\mathbb{R}} f_n(x)\cdot g(x)\,dx \;=\; 2\cdot g'(0) \qquad\qquad (\text{for differentiable } g \text{ with derivative } g' \text{ in } C^o(\mathbb{R}))$$

[01.4] Show that the closed unit ball in ℓ^2, although *closed* and *bounded*, is *not compact*, by showing it is not *sequentially* compact.

[01.5] Show that the *Hilbert cube*

$$C \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \frac{1}{n}\}$$

is compact. More generally, for any sequence of positive reals ε_n,

$$C(\varepsilon) \;=\; \{(z_1, z_2, \ldots) \in \ell^2 : |z_n| \le \varepsilon_n\}$$

is compact if and only if $\sum_n |\varepsilon_n|^2 < \infty$.

[01.6] Show that a closed interval $[a,b]$ has the expected *Lebesgue outer* measure, namely, $|b-a|$, by showing that the *inf* of $\sum_{j=1}^n |b_n - a_n|$ for all finite open covers $[a,b] \subset \bigcup_{j=1}^n (a_j, b_j)$ is $|b-a|$.

[01.7] Let f be a continuous function on $[0,1]$, with $f(0)=0$ and $f(1)=1$. Show that $\{x : f(x) \in [\frac{1}{4}, \frac{3}{4}]\}$ has positive Lebesgue measure.
