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Examples 02

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[This document is http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-02.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 09 Oct 2017.

[02.1] Show that ℓ^2 is *complete* as a metric space.

[02.2] Show that the characteristic function χ_E of a measurable set E is measurable.

[02.3] Show that the product of two $\mathbb{R}$-valued measurable functions on $\mathbb{R}$ is measurable.

[02.4] Use Urysohn's lemma to prove that $C^o[a,b]$ is dense in $L^1[a,b]$.

[02.5] Comparing L^p spaces: let $1 \leq p, p' < \infty$. When is $L^p[a,b] \subset L^{p'}[a,b]$ for finite intervals $[a,b]$ and Lebesgue measure? When is $L^p(\mathbb{R}) \subset L^{p'}(\mathbb{R})$? When is $\ell^p \subset \ell^{p'}$?

[02.6] For positive real numbers $w_1, \ldots, w_n$ such that $\sum_i w_i = 1$, and for positive real numbers $a_1, \ldots, a_n$, show that

$$a_1^{w_1} \ldots a_n^{w_n} \;\leq\; w_1 a_1 + \ldots + w_n a_n$$

[02.7] In ℓ^2, show that the point in the closed unit ball closest to a point v *not* inside that ball is $v/|v|_{\ell^2}$.

[02.8] For a measurable set $E \subset [0, 2\pi]$, show that

$$\lim_{n\to\infty} \int_E \cos nx \; dx \;=\; 0 \;=\; \lim_{n\to\infty} \int_E \sin nx \; dx$$

[02.9] One form of the *sawtooth* function is $f(x) = x - \pi$ on $[0, 2\pi]$. Compute the Fourier coefficients $\widehat{f}(n)$. Write out the conclusion of Plancherel-Parseval's theorem for this function.

[02.10] For fixed $y \in [0, 2\pi]$, show that there is *no* $f_y \in L^2[0, 2\pi]$ so that $\langle g, f_y \rangle = g(y)$ for all $g \in L^2[0, 2\pi]$.
