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Examples 03

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-03.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, 18 Oct 2017.

[03.1] Show that every vector subspace of $\mathbb{R}^n$ and/or $\mathbb{C}^n$ is (topologically) *closed*.

[03.2] For a subspace W of a Hilbert space V, show that $(W^\perp)^\perp$ is the closure of the subspace W in V.

[03.3] Show that for $0 < x < 1$

$$\sum_{n\geq 1} \frac{\sin 2\pi nx}{n} \;=\; \pi \cdot (\tfrac{1}{2} - x)$$

[03.4] Let $c_1, c_2, \ldots$ be positive real, converging monotonically to 0. For $0 < x < 1$, prove that $\sum_{n\geq 0} c_n\, e^{2\pi inx}$ *converges pointwise.*

[03.5] Show that the sup-norm completion of the space $C^o_c(\mathbb{R})$ of compactly-supported continuous functions is the space $C^o_o(\mathbb{R})$ of continuous functions going to 0 at infinity. An analogous assertion and argument should hold for any topological space in place of $\mathbb{R}$.

[03.6] Compute $\displaystyle\int_{\mathbb{R}} \Big(\frac{\sin x}{x}\Big)^2\, dx$. (*Hint:* use Plancherel.)

[03.7] For $f \in L^2(\mathbb{R})$ and $t \in \mathbb{R}$, show that there is a constant C (depending on f) such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\, dx\Big| \;<\; C \cdot \sqrt{\delta}$$

Formulate and prove the corresponding assertion for L^p with $1 < p < \infty$.

[03.8] For $f \in L^1(\mathbb{R})$ and $t \in \mathbb{R}$, show that, given $\varepsilon > 0$, there is $\delta > 0$ such that

$$\Big|\int_{t-\delta}^{t+\delta} f(x)\, dx\Big| \;<\; \varepsilon$$

Sharpen the first example to show that

$$\int_{t-\delta}^{t+\delta} f(x)\, dx \;=\; o(\sqrt{\delta}) \qquad\qquad (\text{as } \delta \to 0^+)$$

where Landau's little-o notation is that $f(x) = o(g(x))$ as $x \to a$ when $\lim_{x\to a} f(x)/g(x) = 0$.
