(*October 20, 2017*)

# Examples 04

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-04.pdf]

For feedback on these examples, please get your write-ups to me by Monday, 30 Oct 2017.

**[04.1]** Fix $x_o \in [a,b]$. Show that $\lambda(f) = f(x_o)$ is a continuous linear functional on $C^o[a,b]$.

**[04.2]** Prove that Cesaro summation

$$b_1 = \frac{a_1}{1},\;\; b_2 = \frac{a_1+a_2}{2},\;\; b_3 = \frac{a_1+a_2+a_3}{3},\ldots$$

converts every convergent sequence $a_1, a_2, \ldots$ to a convergent sequence $b_1, b_2, \ldots$ with the same limit.

**[04.3]** (Collecting Fourier transform pairs...) Compute the Fourier transforms of

$$\chi_{[a,b]} \qquad\qquad e^{-\pi x^2} \qquad\qquad f(x) = \begin{cases} e^{-x} & (\text{for } x > 0) \\ 0 & (\text{for } x \le 0) \end{cases}$$

**[04.4]** Show that $\chi_{[a,b]} * \chi_{[c,d]}$ is a piecewise-line function, and express it explicitly.

**[04.5]** Evaluate the *Borwein integral*

$$\int_{\mathbb{R}} \frac{\sin x}{x} \cdot \frac{\sin x/3}{x/3} \cdot \frac{\sin x/5}{x/5}\; dx$$

**[04.6]** Compute $e^{-\pi x^2} * e^{-\pi x^2}$ and $\dfrac{\sin x}{x} * \dfrac{\sin x}{x}$. (Be careful what you say: $\frac{\sin x}{x}$ is not in $L^1(\mathbb{R})$.)

**[04.7]** Prove that every $f \in C^o_c(\mathbb{R})$ can be uniformly approximated (in sup norm) arbitrarily well by *superpositions of translates of Gaussians*: given $\varepsilon > 0$, there is $\varphi \in C^o_c(\mathbb{R})$ and sufficiently large $n$ such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot e^{-\pi n^2(\xi - x)^2}\; d\xi \Big| \;<\; \varepsilon$$

**[04.8]** Prove that, for given $f \in C^o_c(\mathbb{R})$, given $\varepsilon > 0$, there is sufficiently large $n$ and a function $\varphi \in C^o_c(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} \Big| f(x) - \int_{\mathbb{R}} \varphi(\xi) \cdot \frac{(\sin n(x-\xi))^2}{(x-\xi)^2}\; d\xi \Big| \;<\; \varepsilon$$

**[04.9]** Show that the *principal value* functional

$$f \;\longrightarrow\; PV \int_{\mathbb{R}} \frac{f(x)}{x}\; dx \;=\; \lim_{\varepsilon\to 0} \Big( \int_{-\infty}^{-\varepsilon} \frac{f(x)}{x}\; dx + \int_{\varepsilon}^{\infty} \frac{f(x)}{x}\; dx \Big)$$

is equal to

$$-\int_{\mathbb{R}} f'(x) \cdot \log|x| \, dx$$

for $f$ *continuously* differentiable near 0, with $f \in L^2(\mathbb{R})$, $f(x) \to 0$ as $|x| \to \infty$, and $|f'(x)| \ll \frac{1}{1+x^2}$ for easy convergence.

**[04.10]** Let $\psi_n(x) = e^{2\pi i n x}$. Let $\delta_{\mathbb{Z}}$ be the *Dirac comb*, that is, a periodic version of Dirac's $\delta$, describable as having Fourier series

$$\delta_{\mathbb{Z}} \;=\; \sum_{n\in\mathbb{Z}} 1 \cdot \psi_n \qquad \text{(converging in } H^{-1}(\mathbb{T}) \text{ or even } H^{-\frac{1}{2}-\varepsilon}(\mathbb{T}) \text{ for all } \varepsilon > 0)$$

With $\lambda \notin \mathbb{R}$, show that the differential equation

$$u'' - \lambda \cdot u \;=\; \delta_{\mathbb{Z}}$$

has a periodic solution $u \in H^{\frac{3}{2}-\varepsilon}(\mathbb{T}) \subset C^o(\mathbb{T})$, using Fourier series, *by division*. Show that the equation $v'' - \lambda v = f$ is solved by

$$v \;=\; \int_{\mathbb{T}} u(x-t)\, f(t)\, dt \;=\; \int_0^1 u(x-t)\, f(t)\, dt$$

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