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Examples 05

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-05.pdf]

For feedback on these examples, please get your write-ups to me by Wednesday, 15 Nov 2017.

[05.1] Give a *persuasive* proof that the function

$$f(x) \;=\; \begin{cases} 0 & (\text{for } x \le 0) \\ e^{-1/x} & (\text{for } x > 0) \end{cases}$$

is infinitely differentiable at 0. Use this kind of construction to make a *smooth step function*: 0 for $x \le 0$ and 1 for $x \ge 1$, and goes monotonically from 0 to 1 in the interval $[0,1]$. Use this to construct a *family of smooth cut-off functions* $\{f_n : n = 1,2,3,\ldots\}$: for each n, $f_n(x) = 1$ for $x \in [-n,n]$, $f_n(x) = 0$ for $x \notin [-(n+1), n+1]$, and f_n goes monotonically from 0 to 1 in $[-(n+1), -n]$ and monotonically from 1 to 0 in $[n, n+1]$.

[05.2] With $g(x) = f(x + x_o)$, express $\widehat{g}$ in terms of $\widehat{f}$, first for $f \in \mathscr{S}(\mathbb{R}^n)$, then for $f \in \mathscr{S}(\mathbb{R}^n)^*$.

[05.3] Let V be a vector space, with norms $|\cdot|_1$ and $|\cdot|_2$. Suppose that $|v|_2 \ge |v|_1$ for all $v \in V$. Show that the identity map $i : V \to V$ is continuous, where the source is given the $|\cdot|_2$ topology and the target is given the $|\cdot|_1$ topology. Show that if a sequence $\{v_n\}$ in V is $|\cdot|_2$ Cauchy, then it is $|\cdot|_1$-Cauchy. Let V_j be the completion of V with respect to the metric $|v - v'|_j$. Show that we can *extend i by continuity* to a continuous linear map $I : V_2 \to V_1$, that is, by

$$I(V_2\text{-limit of } V_2\text{-Cauchy sequence } \{v_n\}) \;=\; V_1\text{-limit of } \{v_n\}$$

[05.4] Solve $-u'' + u = \delta$ on $\mathbb{R}$. (*Hint:* use Fourier transform, and grant that $\widehat{\delta} = 1$.)

[05.5] Show that $u'' = \delta_{\mathbb{Z}}$ has no solution on the circle $\mathbb{T}$. (*Hint:* Use Fourier series, granting the Fourier expansion of $\delta_{\mathbb{Z}}$.) Show that $u'' = \delta_{\mathbb{Z}} - 1$ *does* have a solution.

[05.6] On the circle $\mathbb{T}$, show that $u'' = f$ has a unique solution for all $f \in L^2(\mathbb{T})$ orthogonal to the constant function 1.

[05.7] The sawtooth function is first defined on $[0,1)$ by $\sigma(x) = x - \frac{1}{2}$, and then extended to $\mathbb{R}$ by periodicity so that $\sigma(x+n) = \sigma(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$. After recalling its Fourier expansion, describe the derivatives σ' and σ'' of σ.

[05.8] Show that $e^{-\varepsilon\pi x^2} \to 1$ as $\varepsilon \to 0^+$ in the $\mathscr{S}^*$ topology. Compute the Fourier transforms of the functions $e^{-\varepsilon\pi x^2}$, and show that they go to δ in the $\mathscr{S}^*$ topology. Obtain, again, as a corollary, the fact that $\widehat{1} = \delta$ (extended Fourier transform).

[05.9] Compute $\widehat{\cos x}$. (*Hint:* write $\cos x$ in terms of complex exponentials, and observe that these complex exponentials are the Fourier transforms of certain translates of δ.)

[05.10] Smooth functions $f \in \mathcal{E}$ act on distributions $u \in \mathcal{D}(\mathbb{R})^*$ by a dualized form of pointwise multiplication: $(f \cdot u)(\varphi) = u(f\varphi)$ for $\varphi \in \mathcal{D}(\mathbb{R})$. Show that if $x \cdot u = 0$, then u is *supported at* 0, in the sense that for $\varphi \in \mathcal{D}$ with $\operatorname{spt}\varphi \not\ni 0$, necessarily $u(\varphi) = 0$. Thus, by the theorem classifying such distributions, u is a linear combination of δ and its derivatives. Show that in fact $x \cdot u = 0$ implies that u is a multiple of δ itself.
