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Examples 06

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-06.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 01 December 2017.

[06.1] Given f in the Schwartz space \mathcal{S} , show that there is $F \in \mathcal{S}$ with $F' = f$ if and only if $\int_{\mathbb{R}} f = 0$.

[06.2] Let $u(x) = e^x \cdot \sin(e^x)$. Explain in what sense the integral $\int_{\mathbb{R}} f(x) u(x) dx$ converges for every $f \in \mathcal{S}$.

[06.3] Show that $\sin(nx) \rightarrow 0$ in the \mathcal{S}' -topology as $n \rightarrow +\infty$.

[06.4] Let $-\infty < a < b < c < +\infty$, and

$$f(x) = \begin{cases} 0 & (\text{for } x < a) \\ A & (\text{for } a < x < b) \\ B & (\text{for } b < x < c) \\ 0 & (\text{for } c < x) \end{cases}$$

Show that (extended) $\frac{d}{dx} f = A\delta_a + (B - A)\delta_b - B\delta_c$.

[06.5] Show that the principal value functional $u(\varphi) = P.V. \int_{\mathbb{R}} \frac{\varphi(x)}{x} dx$ is in \mathcal{S}' and satisfies $x \cdot u = 1$.

[06.6] Compute the Fourier transform of the sign function

$$\text{sgn}(x) = \begin{cases} 1 & (\text{for } x > 0) \\ -1 & (\text{for } x < 0) \end{cases}$$

Hint: $\frac{d}{dx} \text{sgn} = 2\delta$. Since Fourier transform converts d/dx to multiplication by $2\pi ix$, this implies that $(2\pi i)x \cdot \widehat{\text{sgn}} = 2\widehat{\delta} = 2$. Thus, $(\pi i)x \cdot \widehat{\text{sgn}} = 1$.

[06.7] Show that $x\delta' = -\delta$ on \mathbb{R} . On \mathbb{R}^n , show that $x_i\delta = 0$.

[06.8] On \mathbb{R}^n , show that $r^2 \cdot \Delta\delta = 2n \cdot \delta$.

[06.9] On \mathbb{R}^2 , compute the Fourier transform of $(x \pm iy)^n \cdot e^{-\pi(x^2+y^2)}$ for $n = 0, 1, 2, \dots$ (*Hint:* Re-express things, including Fourier transform, in terms of $z = x + iy$ and $\bar{z} = x - iy$, $w = u + iv$, and $\bar{w} = u - iv$.)
