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Examples 07

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[This document is
http://www.math.umn.edu/~garrett/m/real/examples_2017-18/real-ex-07.pdf]

For feedback on these examples, please get your write-ups to me by Friday, 16 February 2018.

[07.1] Compute the Fourier transform of $|x|$ on $\mathbb{R}$. (Hint: its second derivative is 2δ.)

[07.2] (*Trace Theorem* $\mathbb{T}^2 \to \mathbb{T}^1$) For $f \in H^s(\mathbb{T}^2)$ with $s > \frac{1}{2}$, show that $f\Big|_{\mathbb{T}\times\{0\}} \in H^{s-\frac{1}{2}}(\mathbb{T})$.

[07.3] Let $\psi_\xi(x) = e^{2\pi i\xi\cdot x}$. Tell in what useful sense $\int_{\mathbb{R}^n} 1 \cdot \psi_\xi \, d\xi$ converges.

[07.4] Show that there exists $f \in C^o(\mathbb{R}^n)$ and $0 \le k \in \mathbb{Z}$ such that $(1-\Delta)^k f = \delta$.

[07.5] Show that the characteristic function of an interval is in $H^{\frac{1}{2}-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^{\frac{1}{2}}(\mathbb{R})$.

[07.6] Show that $f(x) = e^{-|x|}$ is in $H^{1-\varepsilon}(\mathbb{R})$ for every $\varepsilon > 0$, but is *not* in $H^1(\mathbb{R})$.

[07.7] Recall the argument that $\delta \in H^{-\frac{n}{2}-\varepsilon}(\mathbb{R}^n)$ for every ε, but is *not* in $H^{-\frac{n}{2}}(\mathbb{R}^n)$.

[07.8] Let u be a distribution on $\mathbb{R}$. Show that $\delta * u = u$ and $\delta' * u = u'$.

[07.9] For compactly supported distributions u, v, show that $(u*v)' = u' * v = u * v'$.

[07.10] Let H be the Heaviside step function (with $H' = \delta$). Let 1 denote the identically-one function. Verify that $(1 * \delta') * H = 0$, while $1 * (\delta' * H) = 1$, so *associativity fails*:

$$(1 * \delta') * H \;=\; 0 \;\neq\; 1 \;=\; 1 * (\delta' * H)$$

(This is not a pathology, because there is no purposeful definition of convolution involving two or more general not-compactly-supported distributions.)

[07.11] (*) Show that the functional on $f \in \mathcal{D}(\mathbb{R}^2)$ given by integrating around the unit circle

$$u(f) \;=\; \int_0^{2\pi} f(\cos\theta,\; \sin\theta)\; d\theta$$

is in $H^{-\frac{1}{2}-\varepsilon}(\mathbb{R}^2)$ for every $\varepsilon > 0$.
